Ramanujan J (2017) 42:777–781
DOI 10.1007/s11139-015-9763-z

ERRATUM

Erratum and Addendum to: Rediscovery of Malmsten's integrals, their evaluation by contour integration methods and some related results [Ramanujan J. (2014), 35:21–110]

Iaroslav V. Blagouchine[1,2]

Published online: 25 October 2016

Erratum to: Ramanujan J (2014) 35:21–110 DOI 10.1007/s11139-013-9528-5

Addendum to Section 2.2

The historical analysis of functional Eqs. (20)–(22) on pp. 35–37 is far from exhaustive. In order to give a larger vision of this subject, several complimentary remarks may be needed.

First, on p. 37, lines 1–5, the text "By the way, the above reflection formula (21) for $L(s)$ was also obtained by Oscar Schlömilch; in 1849 he presented it as an exercise for students [55], and then, in 1858, he published the proof [56]. Yet, it should be recalled that an analog of formula (20) for the alternating…" may be replaced by the following one: "By the way, between 1849 and 1858, the above reflection formula for $L(s)$ was also obtained by several other mathematicians, including Oscar Schlömilch [55, 56], Gotthold Eisenstein [73, 84], and Thomas Clausen [75].[1] Yet, it should be noted that formula (20) itself was rigorously proved by Kinkelin a year before Riemann [79, p. 100], [78], and its analog for the alternating…"

[1] In 1849, Schlömilch presented the theorem as an exercise for students [55]. In 1858, Clausen [75] published the proof to this exercise. The same year, Schlömilch published his own proof [56]. Eisenstein did not publish the proof, but left some drafts dating back to 1849, see e.g. [73, 84].

The online version of the original article can be found under doi:10.1007/s11139-013-9528-5.

✉ Iaroslav V. Blagouchine
iaroslav.blagouchine@univ-tln.fr; iaroslav.blagouchine@pdmi.ras.ru

1 University of Toulon, La Garde, France

2 Steklov Institute of Mathematics at St. Petersburg, St. Petersburg, Russia

Second, on p. 37, in formula (22), the confusing part “$n = 1, 2, 3, \dots$” should be replaced by “$n \in \mathbb{R}$”. In fact, by comparing values of $\eta(1-n)$ to $\eta(n)$ at positive integers and by noticing that both of them contain the same Bernoulli numbers, Euler deduced Eq. (22). After that, he carried out a number of complimentary verifications, which suggested that Eq. (22) should hold not only for integer values of n, but also for fractional and continuous values of n. Whence, he conjectured that (22) should be true for any value of argument, including continuous values of n. In particular, on p. 94 of [20], Euler wrote: *“Par cette raison j’hazarderai la conjecture suivante, que quelque soit l’exposant n, cette equation a toujours lieu :*

$$\frac{1 - 2^{n-1} + 3^{n-1} - 4^{n-1} + 5^{n-1} - 6^{n-1} + \dots}{1 - 2^{-n} + 3^{-n} - 4^{-n} + 5^{-n} - 6^{-n} + \dots} = \frac{-1 \cdot 2 \cdot 3 \cdots (n-1)\,(2^n - 1)}{(2^{n-1} - 1)\,\pi} \cos\frac{\pi n}{2} \text{.”}$$

The latter is our Eq. (22) and is also equivalent to (20). By the way, Hardy’s exposition of Euler’s achievements, which we cited in footnote 15, Ref. [31, pp. 23–26], is also far from exhaustive. For instance, Hardy did not mention the fact that Euler have conjectured that formula (22) remains true for any value of n. Moreover, Hardy says that it was comparatively recently that it was observed, first by Cahen and then by Landau, that the reflection formulas for $L(s)$ and $\eta(s)$ both stand in Euler’s paper written in 1749. This is, however, not true. Thus, Malmsten in 1846 remarked [40, p. 18] that (21) were obtained by Euler by induction, and Hardy cited this work of Malmsten. Unfortunately, Hardy did not notice that Malmsten also quoted Euler.

Third, on p. 37, after the last sentence in the first paragraph ending by “…requires the notion of analytic continuation.”, the following footnote may be added

> “An alternative historical analysis of functional Eqs. (20)–(21) in the context of contributions of various authors may be found in [85], [31, p. 23], [84], [82, p. 4], [78, p. 193], [74, pp. 326–328], [83, p. 298], [73]. Note, however, that Butzer et al.’s statement [74, p. 328] “Malmstén included the functional equation without proof” is rather incorrect. Thus, André Weil [84, p. 8] points out that “Malmstén included the proof in a long paper written in May 1846”. Moreover, our investigations show that this proof was not only included in his paper [41] written in 1846, but also was present in an earlier work [40] published in 1842. By the way, Malmsten remarked that reflection formulas of such kind were first announced by Euler in 1749, the fact which was not mentioned by Schlömilch [55, 56], nor by Clausen [75], nor by Kinkelin [79], nor by Riemann [54].”

Addendum to Section 4.1.2, Exercise no. 18

Results of this exercise also permit to evaluate some very curious integrals containing $\cos\ln\ln x$ and $\sin\ln\ln x$ in the numerator. Putting in the last unnumbered equation in

Exercise no. 18 on p. 66 $a = i\alpha$, $\alpha \in \mathbb{R}$, and $b = 1$, we have

$$\begin{aligned}
\int_0^\infty \frac{\cos(\alpha \ln x)}{\mathrm{ch} x}\,\mathrm{d}x &= 2\int_1^\infty \frac{\cos(\alpha \ln\ln x)}{1+x^2}\,\mathrm{d}x \\
&= -\alpha \mathrm{Im}\left[\frac{\Gamma(i\alpha)}{2^{2i\alpha}}\left\{\zeta\big(1+i\alpha, 1/4\big) - 2^{i\alpha}\big(2^{1+i\alpha}-1\big)\zeta(1+i\alpha)\right\}\right], \\
\int_0^\infty \frac{\sin(\alpha \ln x)}{\mathrm{ch} x}\,\mathrm{d}x &= 2\int_1^\infty \frac{\sin(\alpha \ln\ln x)}{1+x^2}\,\mathrm{d}x \\
&= \alpha \mathrm{Re}\left[\frac{\Gamma(i\alpha)}{2^{2i\alpha}}\left\{\zeta\big(1+i\alpha, 1/4\big) - 2^{i\alpha}\big(2^{1+i\alpha}-1\big)\zeta(1+i\alpha)\right\}\right],
\end{aligned}$$

These integrals are, in some sense, complimentary to basic Malmsten's integrals (1)–(2), which were evaluated in Sect. 3.4 and 4.1.2, no. 18-g, and readily permit to evaluate integrals

$$\int_0^\infty \frac{\ln^n x}{\mathrm{ch} x}\,\mathrm{d}x = 2\int_1^\infty \frac{\ln^n \ln x}{1+x^2}\,\mathrm{d}x = 2\int_0^1 \frac{\ln^n \ln \frac{1}{x}}{1+x^2}\,\mathrm{d}x\,, \qquad n = 1, 2, 3, \ldots$$

in terms of Stieltjes constants (first two such expressions were given in Exercises no. 18-g and 18-h). It is also interesting that right parts of both expressions contain $\zeta(1+i\alpha)$, which was found to be connected with the nontrivial zeros of the ζ-function.[2]

Addendum to Section 4.2, Exercise no. 29

In right parts of formulas (d)–(g), it may be more preferable to have $\ln(1+\sqrt{2})$ rather than $\ln(2 \pm \sqrt{2})$

$$\begin{aligned}
\text{(d)}\quad \int_0^1 \frac{\ln\ln \frac{1}{x}}{1+\sqrt{2}\,x+x^2}\,\mathrm{d}x &= \int_1^\infty \frac{\ln\ln x}{1+\sqrt{2}\,x+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{4\sqrt{2}}\left\{5\ln\pi + 4\ln 2 - 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{3}{8}\right)\right\}, \\
\text{(e)}\quad \int_0^1 \frac{\ln\ln \frac{1}{x}}{1-\sqrt{2}\,x+x^2}\,\mathrm{d}x &= \int_1^\infty \frac{\ln\ln x}{1-\sqrt{2}\,x+x^2}\,\mathrm{d}x \\
&= \frac{\pi}{4\sqrt{2}}\left\{7\ln\pi + 6\ln 2 + 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{1}{8}\right)\right\},
\end{aligned}$$

[2] Estimation of $\left|\zeta(1+i\alpha)\right|$ was found to be connected with $\mathrm{Re}\rho$, where ρ are the zeros of $\zeta(s)$ in the critical strip $0 \leqslant \mathrm{Re}s \leqslant 1$, see e.g. [80, p. 128].

(f) $$\int_0^1 \frac{x \ln\ln \frac{1}{x}}{1+\sqrt{2}\,x^2+x^4}\,\mathrm{d}x = \int_1^\infty \frac{x \ln\ln x}{1+\sqrt{2}\,x^2+x^4}\,\mathrm{d}x$$
$$= \frac{\pi}{8\sqrt{2}}\left\{5\ln\pi + 3\ln 2 - 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{3}{8}\right)\right\},$$

(g) $$\int_0^1 \frac{x \ln\ln \frac{1}{x}}{1-\sqrt{2}\,x^2+x^4}\,\mathrm{d}x = \int_1^\infty \frac{x \ln\ln x}{1-\sqrt{2}\,x^2+x^4}\,\mathrm{d}x$$
$$= \frac{\pi}{8\sqrt{2}}\left\{7\ln\pi + 3\ln 2 + 2\ln(1+\sqrt{2}) - 8\ln\Gamma\left(\frac{1}{8}\right)\right\}.$$

Addendum to Section 4.5, Exercise no. 62-b

On p. 96, in Exercise no. 62-b, in the unnumbered formula after Eq. (56), in the first line the last term

$$-\frac{1}{2}\sum_{l=1}^{n-1}\alpha_{l,n}\varsigma_{l,n}$$

may be removed. Strictly speaking, the actual expression for $\sum \Upsilon_{k,n}\ln\Gamma\left(\frac{k}{n}\right)$ is correct. However, because of the symmetry, the function $\varsigma_{l,n}$ identically vanishes for any integer $l = 1, 2, \ldots, n-1$, and hence, so does the last term in the first line of this formula.

Some minor corrections and additions

- p. 42, line 20: "has no branch points." should read "has no branch points except at poles of $\Gamma(z)$."
- p. 42, line 27: "points at all, which allows" should read "points at all in the right half–plane, which allows".
- p. 66, in *Nota Bene* of exercise no. 19: "derived by Malmsten in [41, unnumbered" should read "derived by Malmsten in [40, p. 24, Eq. (37)], [41, unnumbered".
- p. 68, first line: "no. 21-e" should read "no. 21-d".
- p. 73, last line, "$|\mathrm{Re} r| < 2\pi$" should read "$|\mathrm{Im} r| < 2\pi$".
- p. 82, line 7, "no. 39-c is given" should read "no. 39-e is given".
- p. 83, exercise no. 40: formula given in exercise no. 40-b, as well as formula (55), were also obtained by Nørlund in [81, p. 107].
- p. 97, footnote 40 "formula (c) was" should read "formula (b.2) was".[3]

[3] It may also noted that in a later work [72, pp. 542–543], we showed that (b.1), which is a shifted version of (b.2), was already known to Malmsten in 1846.

- p. 100, exercise no. 64: closed-form expressions equivalent to those we gave for $\gamma_1(1/2)$, $\gamma_1(1/4)$, $\gamma_1(3/4)$ and $\gamma_1(1/3)$ were also obtained by Connon in [76, pp. 1, 50, 53, 54–55], [77, pp. 17–18].

References

72. Blagouchine, Ia.V.: A theorem for the closed–form evaluation of the first generalized Stieltjes constant at rational arguments and some related summations, J. Number Theory **148**, 537–592 and **151**, 276–277, arXiv:1401.3724 (2015)
73. Bombieri, E.: The classical theory of zeta and L-functions. Milan J. Math. **78**(1), 11–59 (2010)
74. Butzer, P.M., Jansche, S.: A direct approach to the Mellin transform. J. Fourier Anal. Appl. **3**(4), 325–376 (1997)
75. Clausen, T.: Beweis des von Schlömilch Archiv Bd. XII. No. XXXV. aufgestellten Lehssatzes; — über die Ableitung des Differentials von log Γx; und — über eine allgemeine Aufgabe über die Funktionen von Abel, Grunert Archiv der Mathematik und Physik, vol. XXX, pp. 166–170 (1858)
76. Connon, D.F.: Some series and integrals involving the Riemann zeta function, binomial coefficients and the harmonic numbers. vol. II(b), arXiv:0710.4024 (2007)
77. Connon, D.F.: The difference between two Stieltjes constants, arXiv:0906.0277 (2009)
78. Dutka, J.: On the summation of some divergent series of Euler and the zeta functions. Arch. Hist. Exact Sci. **50**(2), 187–200 (1996)
79. Kinkelin, H.: Ueber einige unendliche Reihen. Mitteilungen der Naturforschenden Gesellschaft in Bern, no. **419–420**, 89–104 (1858)
80. Linnik, Y.V.: On Hilbert’s eighth problem. In: Alexandrov, P.S. (ed.) Hilbert’s Problems (in Russian), pp. 128–130. Nauka, Moscow (1969)
81. Nörlund, N.E.: Vorlesungen über Differenzenrechnung. Springer, Berlin (1924)
82. Patterson, S.J.: An Introduction to the Theory of the Riemann Zeta-function. Cambridge Studies in Advanced Mathematics, vol. 14. Cambridge University Press, Cambridge (1988)
83. Srinivasan, G.K.: The gamma function: an eclectic tour. Am. Math. Mon. **114**(4), 297–315 (2007)
84. Weil, A.: Prehistory of the zeta-function. In: Aubert, K., Bombieri, E., Goldfeld, D. (eds.) Number Theory, Trace Formulas and Discrete Groups, pp. 1–9. Academic Press, London (1989)
85. Wieleitner, H.: Geschichte der Mathematik (in 2 vols.), Berlin (1922–1923)