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Expansions of generalized Euler's constants into the series of polynomials in π^{-2} and into the formal enveloping series with rational coefficients only

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A B S T R A C T

In this work, two new series expansions for generalized Euler's constants (Stieltjes constants) γ_m are obtained. The first expansion involves Stirling numbers of the first kind, contains polynomials in π^{-2} with rational coefficients and converges slightly better than Euler's series $\sum n^{-2}$. The second expansion is a semi-convergent series with rational coefficients only. This expansion is particularly simple and involves Bernoulli numbers with a non-linear combination of generalized harmonic numbers. It also permits to derive an interesting estimation for generalized Euler's constants, which is more accurate than several well-known estimations. Finally, in Appendix A, the reader will also find two simple integral definitions for the Stirling numbers of the first kind, as well an upper bound for them.

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1. Introduction and notations

1.1. Introduction

The ζ-function, which is usually introduced via one of the following series,

$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}\,, & \operatorname{Re} s > 1 \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}\,, & \operatorname{Re} s > 0\,, \quad s \neq 1 \end{cases} \tag{1}$$

is of fundamental and long-standing importance in modern analysis, number theory, theory of special functions and in a variety other fields. It is well known that $\zeta(s)$ is meromorphic on the entire complex s-plane and that it has one simple pole at $s = 1$ with residue 1. Its expansion in the Laurent series in a neighbourhood of $s = 1$ is usually written the following form

$$\zeta(s) = \frac{1}{s-1} + \sum_{m=0}^{\infty} \frac{(-1)^m (s-1)^m}{m!} \gamma_m\,, \qquad s \neq 1, \tag{2}$$

where coefficients γ_m, appearing in the regular part of expansion (2), are called *generalized Euler's constants* or *Stieltjes constants*, both names being in use.[2,3] Series (2) is the standard definition for γ_m. Alternatively, these constants may be also defined via the following limit

$$\gamma_m = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} \frac{\ln^m k}{k} - \frac{\ln^{m+1} n}{m+1} \right\}, \quad m = 0, 1, 2, \ldots \tag{3}$$

The equivalence between definitions (2) and (3) was demonstrated by various authors, including Adolf Pilz [69], Thomas Stieltjes, Charles Hermite [1, vol. I, letter 71 and following], Johan Jensen [87,89], Jérôme Franel [56], Jørgen P. Gram [69], Godfrey H. Hardy [73], Srinivasa Ramanujan [2], William E. Briggs, S. Chowla [24] and many others, see e.g. [16,176,84,128]. It is well known that $\gamma_0 = \gamma$ Euler's constant, see e.g. [128],

[2] The definition of Stieltjes constants accordingly to formula (2) is due to Godfrey H. Hardy. Definitions, introduced by Thomas Stieltjes and Charles Hermite between 1882–1884, did not contain coefficients $(-1)^m$ and $m!$ In fact, use of these factors is not well justified; notwithstanding, Hardy's form (2) is largely accepted and is more frequently encountered in modern literature. For more details, see [1, vol. I, letter 71 and following], [110, p. 562], [19, pp. 538–539].

[3] Some authors use the name *generalized Euler's constants* for other constants, which were conceptually introduced and studied by Briggs in 1961 [23] and Lehmer in 1975 [114]. They were subsequently rediscovered in various (usually slightly different) forms by several authors, see e.g. [173,140,190]. Further generalization of both, generalized Euler's constants defined accordingly to (2) and generalized Euler's constants introduced by Briggs and Lehmer, was done by Dilcher in [49].

[19, Eq. (14)]. Higher generalized Euler's constants are not known to be related to the "standard" mathematical constants, nor to the "classic" functions of analysis.

In our recent work [18], we obtained two interesting series representations for the logarithm of the Γ-function containing Stirling numbers of the first kind $S_1(n,k)$

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + \\ + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l)!\cdot|S_1(n,2l+1)|}{(2\pi z)^{2l+1}} \tag{4}$$

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln\left(z-\frac{1}{2}\right) - z + \frac{1}{2} + \frac{1}{2}\ln 2\pi - \\ - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\cdot n!}\sum_{l=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^l\,\frac{(2l)!\cdot(2^{2l+1}-1)\cdot|S_1(n,2l+1)|}{(4\pi)^{2l+1}\cdot\left(z-\frac{1}{2}\right)^{2l+1}} \tag{5}$$

as well as their analogs for the polygamma functions $\Psi_k(z)$.[4] The present paper is a continuation of this previous work, in which we show that the use of a similar technique permits to derive two new series expansions for generalized Euler's constants γ_m, both series involving Stirling numbers of the first kind. The first series is convergent and contains polynomials in π^{-2} with rational coefficients (the latter involves Stirling numbers of the first kind). From this series, by a formal procedure, we deduce the second expansion, which is semi-convergent and contains rational terms only. This expansion is particularly simple and involves only Bernoulli numbers and a non-linear combination of generalized harmonic numbers. Convergence analysis of discovered series shows that the former converges slightly better than Euler's series $\sum n^{-2}$, in a rough approximation at the same rate as

$$\sum_{n=3}^{\infty}\frac{\ln^m \ln n}{n^2\ln^2 n}\,, \qquad m=0,1,2,\ldots$$

The latter series diverges very quickly, approximately as

$$\sum_{n=2}^{\infty}(-1)^{n-1}\frac{\ln^m n}{\sqrt{n}}\left(\frac{n}{\pi e}\right)^{2n}, \qquad m=0,1,2,\ldots$$

1.2. Notations and some definitions

Throughout the manuscript, following abbreviated notations are used: $\gamma = 0.5772156649\ldots$ for Euler's constant, γ_m for mth generalized Euler's constant (Stieltjes

[4] Both series converge in a part of the right half-plane [18, Fig. 2] at the same rate as $\sum\left(n\ln^m n\right)^{-2}$, where $m=1$ for $\ln\Gamma(z)$ and $\Psi(z)$, $m=2$ for $\Psi_1(z)$ and $\Psi_2(z)$, $m=3$ for $\Psi_3(z)$ and $\Psi_4(z)$, *etc.*

constant) accordingly to their definition (2),[5] $\binom{k}{n}$ denotes the binomial coefficient C_k^n, B_n stands for the nth Bernoulli number,[6] H_n and $H_n^{(s)}$ denote the nth harmonic number and the nth generalized harmonic number of order s

$$H_n \equiv \sum_{k=1}^{n} \frac{1}{k}\,, \qquad\qquad H_n^{(s)} \equiv \sum_{k=1}^{n} \frac{1}{k^s}\,,$$

respectively. Writings $\lfloor x \rfloor$ stands for the integer part of x, $\operatorname{tg} z$ for the tangent of z, $\operatorname{ctg} z$ for the cotangent of z, $\operatorname{ch} z$ for the hyperbolic cosine of z, $\operatorname{sh} z$ for the hyperbolic sine of z, $\operatorname{th} z$ for the hyperbolic tangent of z, $\operatorname{cth} z$ for the hyperbolic cotangent of z. In order to avoid any confusion between compositional inverse and multiplicative inverse, inverse trigonometric and hyperbolic functions are denoted as arccos, arcsin, arctg, ... and not as $\cos^{-1}$, $\sin^{-1}$, tg^{-1}, ... Writings $\Gamma(z)$ and $\zeta(z)$ denote respectively the gamma and the zeta functions of argument z. The Pochhammer symbol $(z)_n$, which is also known as the generalized factorial function, is defined as the rising factorial $(z)_n \equiv z(z+1)(z+2)\cdots(z+n-1) = \Gamma(z+n)/\Gamma(z)$.[7,8] For sufficiently large n, not necessarily integer, the latter can be given by this useful approximation

$$\begin{aligned}(z)_n &= \frac{n^{n+z-\frac{1}{2}}\sqrt{2\pi}}{\Gamma(z)\,e^n}\left\{1+\frac{6z^2-6z+1}{12n}+\frac{36z^4-120z^3+120z^2-36z+1}{288n^2}+O(n^{-3})\right\}\\ &= \frac{n^z\cdot\Gamma(n)}{\Gamma(z)}\left\{1+\frac{z(z-1)}{2n}+\frac{z(z-1)(z-2)(3z-1)}{24n^2}+O(n^{-3})\right\}\end{aligned} \tag{6}$$

which follows from the Stirling formula for the Γ-function.[9] Unsigned (or signless) and signed Stirling numbers of the first kind, which are also known as *factorial coefficients*, are denoted as $|S_1(n,l)|$ and $S_1(n,l)$ respectively (the latter are related to the former as $S_1(n,l) = (-1)^{n\pm l}|S_1(n,l)|$).[10] Because in literature various names, notations and definitions were adopted for the Stirling numbers of the first kind, we specify that we use exactly the same definitions and notation as in [18, Section 2.1], that is to say $|S_1(n,l)|$ and $S_1(n,l)$ are defined as the coefficients in the expansion of rising/falling factorial

---

[5] In particular $\gamma_1 = -0.07281584548\ldots$, $\gamma_2 = -0.009690363192\ldots$, $\gamma_3 = +0.002053834420\ldots$.

[6] In particular $B_0 = +1$, $B_1 = -\frac{1}{2}$, $B_2 = +\frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = +\frac{1}{42}$, $B_7 = 0$, $B_8 = -\frac{1}{30}$, $B_9 = 0$, $B_{10} = +\frac{5}{66}$, $B_{11} = 0$, $B_{12} = -\frac{691}{2730}$, *etc.*, see [3, Tab. 23.2, p. 810], [109, p. 5] or [59, p. 258] for further values. Note also that some authors may use slightly different definitions for the Bernoulli numbers, see e.g. [72, p. 91], [116, pp. 32, 71], [71, p. 19, n° 138] or [11, pp. 3–6].

[7] For nonpositive and complex n, only the latter definition $(z)_n \equiv \Gamma(z+n)/\Gamma(z)$ holds.

[8] Note that some writers (mostly German-speaking) call such a function *faculté analytique* or *Facultät*, see e.g. [157], [158, p. 186], [159, vol. II, p. 12], [72, p. 119], [106]. Other names and notations for $(z)_n$ are briefly discussed in [92, pp. 45–47] and in [68, pp. 47–48].

[9] A simpler variant of the above formula may be found in [177].

[10] There exist more than 50 notations for the Stirling numbers, see e.g. [67], [92, pp. vii–viii, 142, 168], [101, pp. 410–422], [68, Sect. 6.1], and we do not insist on our particular notation, which may seem for certain not properly chosen.

$$
\begin{cases}
\displaystyle\prod_{k=0}^{n-1}(z+k) \;=\; (z)_n \;=\; \frac{\Gamma(z+n)}{\Gamma(z)} \;=\; \sum_{l=1}^{n}|S_1(n,l)|\cdot z^l \;=\; \sum_{l=0}^{\infty}|S_1(n,l)|\cdot z^l \\[4mm]
\displaystyle\prod_{k=0}^{n-1}(z-k) \;=\; (z-n+1)_n \;=\; \frac{\Gamma(z+1)}{\Gamma(z+1-n)} \;=\; \sum_{l=1}^{n}S_1(n,l)\cdot z^l \;=\; \sum_{l=0}^{\infty}S_1(n,l)\cdot z^l
\end{cases}
\tag{7a,b}
$$

respectively, where $z\in\mathbb{C}$ and $n\geqslant 1$. Note that if $l\notin[1,n]$, where l is supposed to be nonnegative, then $S_1(n,l)=0$, except for $S_1(0,0)$ which is set to 1 by convention. Alternatively, the same numbers may be equally defined as the coefficients in the following MacLaurin series

$$
\begin{cases}
\displaystyle (-1)^l\frac{\ln^l(1-z)}{l!} \;=\sum_{n=l}^{\infty}\frac{|S_1(n,l)|}{n!}z^n \;=\sum_{n=0}^{\infty}\frac{|S_1(n,l)|}{n!}z^n\,, \qquad |z|<1\,, \quad l=0,1,2,\ldots \\[4mm]
\displaystyle \frac{\ln^l(1+z)}{l!} \;=\sum_{n=l}^{\infty}\frac{S_1(n,l)}{n!}z^n \;=\sum_{n=0}^{\infty}\frac{S_1(n,l)}{n!}z^n\,, \qquad |z|<1\,, \quad l=0,1,2,\ldots
\end{cases}
\tag{8a,b}
$$

Signed Stirling numbers of the first kind, as we defined them above, may be also given via the following explicit formula

$$
S_1(n,l) \;=\; \frac{(2n-l)!}{(l-1)!}\sum_{k=0}^{n-l}\frac{1}{(n+k)(n-l-k)!(n-l+k)!}\sum_{r=0}^{k}\frac{(-1)^r r^{n-l+k}}{r!(k-r)!} \tag{9}
$$

$l\in[1,n]$, which may be useful for the computation of $S_1(n,l)$ when n is not very large.[11] All three above definitions agree with those adopted by Jordan [92, Chapt. IV], [90,91], Riordan [147, p. 70 *et seq.*], Mitrinović [125], Abramowitz and Stegun [3, n° 24.1.3, p. 824] and many others (moreover, modern CAS, such as *Maple* or *Mathematica*, also share these definitions; in particular `Stirling1(n,l)` in the former and `StirlingS1[n,l]` in the latter correspond to our $S_1(n,l)$).[12] Kronecker symbol (or Kronecker delta) of arguments l and k is denoted by $\delta_{l,k}$ ($\delta_{l,k}=1$ if $l=k$ and $\delta_{l,k}=0$ if $l\neq k$). $\operatorname{Re} z$ and $\operatorname{Im} z$ denote respectively real and imaginary parts of z. Letter i is never used as index and is $\sqrt{-1}$. The writing $\operatorname{res}_{z=a} f(z)$ stands for the residue of the function $f(z)$ at the point $z=a$. Finally, by the relative error between the quantity A and its approximated value B, we mean $(A-B)/A$. Other notations are standard.

[11] From the above definitions, it follows that: $S_1(1,1)=+1$, $S_1(2,1)=-1$, $S_1(2,2)=+1$, $S_1(3,1)=+2$, $S_1(3,2)=-3$, $S_1(3,3)=+1$, $\ldots$, $S_1(8,5)=-1960$, $\ldots$, $S_1(9,3)=+118\,124$, *etc.* Note that there is an error in Stirling's treatise [172]: in the last line in the table on p. 11 [172] the value of $|S_1(9,3)|=118\,124$ and not 105 056. This error has been noted by Jacques Binet [17, p. 231], Charles Tweedie [179, p. 10] and some others (it was also corrected in some translations of [172]).

[12] A quick analysis of several alternative names, notations and definitions may be found in works of Charles Jordan [92, pp. vii–viii, 1 and Chapt. IV], Gould [67,66], and Donald E. Knuth [68, Sect. 6.1], [101, pp. 410–422].

2. A convergent series representation for generalized Euler's constants γ_m involving Stirling numbers and polynomials in π^{-2}

2.1. Derivation of the series expansion

In 1893 Johan Jensen [88,89] by contour integration methods obtained an integral formula for the ζ-function

$$\begin{aligned}\zeta(s) &= \frac{1}{s-1} + \frac{1}{2} + 2\int\limits_0^{\pi/2} \frac{(\cos\theta)^{s-2}\sin s\theta}{e^{2\pi\,\operatorname{tg}\theta}-1}\,d\theta \\ &= \frac{1}{s-1} + \frac{1}{2} + 2\int\limits_0^{\infty} \frac{\sin(s\operatorname{arctg} x)}{(e^{2\pi x}-1)\,(x^2+1)^{s/2}}\,dx \\ &= \frac{1}{s-1} + \frac{1}{2} + \frac{1}{i}\int\limits_0^{\infty} \frac{(1-ix)^{-s}-(1+ix)^{-s}}{e^{2\pi x}-1}\,dx\,, \qquad s\neq 1 \end{aligned} \tag{10}$$

which extends (1) to the entire complex plane except $s = 1$. Expanding the above formula into the Laurent series about $s = 1$ and performing the term-by-term comparison of the derived expansion with the Laurent series (2) yields the following representation for the mth Stieltjes constant

$$\gamma_m = \frac{1}{2}\delta_{m,0} + \frac{1}{i}\int\limits_0^{\infty} \frac{dx}{e^{2\pi x}-1}\left\{\frac{\ln^m(1-ix)}{1-ix} - \frac{\ln^m(1+ix)}{1+ix}\right\}, \qquad m=0,1,2,\ldots \tag{11}$$

which is due to the Jensen and Franel.[13] Making a change of variable in the latter formula $x = -\frac{1}{2\pi}\ln(1-u)$, we have

$$\gamma_m = \frac{1}{2}\delta_{m,0} + \frac{1}{2\pi i}\int\limits_0^1 \left\{\frac{\ln^m\left[1-\dfrac{\ln(1-u)}{2\pi i}\right]}{1-\dfrac{\ln(1-u)}{2\pi i}} - \frac{\ln^m\left[1+\dfrac{\ln(1-u)}{2\pi i}\right]}{1+\dfrac{\ln(1-u)}{2\pi i}}\right\}\frac{du}{u} \tag{12}$$

where $m = 0, 1, 2, \ldots$.

Now, in what follows, we will use a number of basic properties of Stirling numbers, which can be found in an amount sufficient for the present purpose in the following literature: [172,79,106], [112, Book I, part I], [52,155–157], [158, pp. 186–187],

[13] In the explicit form, this integral formula was given by Franel in 1895 [56] (note that the original Franel's formula contained an error and was not valid for $m = 0$). However, it was remarked by Jensen [89] that it can be elementary derived from (10) obtained two years earlier and it is hard to disagree with him. By the way, it is curious that in works of modern authors, see e.g. [42,37], formula (11) is often attributed to Ainsworth and Howell, who discovered it independently much later [8].

[159, vol. II, pp. 23–31], [10,32–34,21,62], [29, p. 129], [92, Chapt. IV], [90,91,132], [133, pp. 67–78], [134,179], [68, Sect. 6.1], [101, pp. 410–422], [40, Chapt. V], [50], [141, Chapt. 4, §3, n° 196–n° 210], [72, p. 60 *et seq.*], [130], [147, p. 70 *et seq.*], [169, vol. 1], [15], [36, Chapt. 8], [3, n° 24.1.3, p. 824], [102, Sect. 21.5-1, p. 824], [13, vol. III, p. 257], [135,171], [44, pp. 91–94], [185, pp. 2862–2865], [11, Chapt. 2], [125,65,67,66,183,29,31], [137, p. 642], [152,61,189,126,14,188,174,80,27,26,83,4,175,70,117,163,164,154,148,149,76, 105,18]. Note that many writers discovered these numbers independently, without realizing that they deal with the Stirling numbers. For this reason, in many sources, these numbers may appear under different names, different notations and even slightly different definitions.[14]

Consider the generating equation for the unsigned Stirling numbers of the first kind, formula (8a). This power series is uniformly and absolutely convergent inside the disk $|z| < 1$. Putting $l + m - 1$ instead of l, multiplying both sides by $(l)_m$ and summing over $l = [1, \infty)$, we obtain for the left side

$$\sum_{l=1}^{\infty}(l)_m\cdot\frac{\big[-\ln(1-z)\big]^{l+m-1}}{(l+m-1)!}=\sum_{l=1}^{\infty}\frac{\big[-\ln(1-z)\big]^{l+m-1}}{(l-1)!}=$$

$$=\big[-\ln(1-z)\big]^{m}\cdot\underbrace{\sum_{l=1}^{\infty}\frac{\big[-\ln(1-z)\big]^{l-1}}{(l-1)!}}_{e^{-\ln(1-z)}}=(-1)^m\cdot\frac{\ln^m(1-z)}{1-z}$$

while the right side of (8a), in virtue of the absolute convergence, becomes

$$\sum_{l=1}^{\infty}(l)_m\cdot\sum_{n=0}^{\infty}\frac{\big|S_1(n,l+m-1)\big|}{n!}\,z^n=\sum_{n=0}^{\infty}\frac{z^n}{n!}\cdot\sum_{l=1}^{n-m+1}(l)_m\cdot\big|S_1(n,l+m-1)\big|$$

$$=m!\cdot\sum_{n=0}^{\infty}\frac{\big|S_1(n+1,m+1)\big|}{n!}\,z^n$$

Whence

$$\frac{\ln^m(1-z)}{1-z}=(-1)^m m!\cdot\sum_{n=0}^{\infty}\frac{\big|S_1(n+1,m+1)\big|}{n!}\,z^n\,,\qquad \begin{array}{l} m=0,1,2,\ldots\\ |z|<1\end{array}\tag{13}$$

[14] Actually, only in the beginning of the XXth century, the name “Stirling numbers” appeared in mathematical literature (mainly, thanks to Thorvald N. Thiele and Niels Nielsen [132,179], [101, p. 416]). Other names for these numbers include: *factorial coefficients*, *faculty’s coefficients* (*Facultätencoefficienten*, *coefficients de la faculté analytique*), *differences of zero* and even *differential coefficients of nothing*. Moreover, the Stirling numbers are also closely connected to the *generalized Bernoulli numbers* $B_n^{(s)}$, also known as *Bernoulli numbers of higher order*, see e.g. [29, p. 129], [65, p. 449], [67, p. 116]; many of their properties may be, therefore, deduced from those of $B_n^{(s)}$.

Writing in the latter $-z$ for z, and then subtracting one from another yields the following series

$$\frac{\ln^m(1-z)}{1-z} - \frac{\ln^m(1+z)}{1+z} = 2(-1)^m m! \cdot \sum_{k=0}^{\infty} \frac{\big|S_1(2k+2,m+1)\big|}{(2k+1)!}\, z^{2k+1} \tag{14}$$

$m = 0, 1, 2, \ldots$, which is absolutely and uniformly convergent in the unit disk $|z| < 1$, and whose coefficients grow logarithmically with k

$$\frac{\big|S_1(2k+2,m+1)\big|}{(2k+1)!} \sim \frac{\ln^m k}{m!}\,, \qquad k\to\infty\,, \qquad m = 0, 1, 2, \ldots \tag{15}$$

in virtue of known asymptotics for the Stirling numbers, see e.g. [91, p. 261], [92, p. 161], [3, n° 24.1.3, p. 824], [188, p. 348, Eq. (8)]. Using formulæ from [40, p. 217], [163, p. 1395], [104, p. 425, Eq. (43)], the law for the formation of first coefficients may be also written in a more simple form

$$\frac{\big|S_1(2k+2,m+1)\big|}{(2k+1)!} = \begin{cases} 1\,, & m=0 \\ H_{2k+1}\,, & m=1 \\ \frac{1}{2}\big\{H_{2k+1}^2 - H_{2k+1}^{(2)}\big\}\,, & m=2 \\ \frac{1}{6}\big\{H_{2k+1}^3 - 3H_{2k+1}H_{2k+1}^{(2)} + 2H_{2k+1}^{(3)}\big\}\,, & m=3 \end{cases} \tag{16}$$

For higher m, values of this coefficient may be similarly reduced to a non-linear combination of the generalized harmonic numbers. Since expansion (14) holds only inside the unit circle, it cannot be directly used for the insertion into Jensen–Franel's integral formula (11). However, if we put in (14) $z = \frac{1}{2\pi i}\ln(1-u)$, we obtain for the right part

$$\begin{aligned} & 2\,(-1)^m m! \cdot \sum_{k=0}^{\infty} \frac{\big|S_1(2k+2,m+1)\big|}{(2\pi i)^{2k+1}} \cdot \underbrace{\frac{\ln^{2k+1}(1-u)}{(2k+1)!}}_{\text{see (8a)}} = \\ & = 2i\,(-1)^m m! \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \big|S_1(2k+2,m+1)\big|}{(2\pi)^{2k+1}} \cdot \sum_{n=1}^{\infty} \frac{\big|S_1(n,2k+1)\big|}{n!}\, u^n \\ & = 2i\,(-1)^m m! \cdot \sum_{n=1}^{\infty} \frac{u^n}{n!} \cdot \sum_{k=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^k \big|S_1(2k+2,m+1)\big| \cdot \big|S_1(n,2k+1)\big|}{(2\pi)^{2k+1}} \end{aligned} \tag{17}$$

Therefore, for $m = 0, 1, 2, \ldots$, we have

$$\frac{1}{2\pi i}\left\{ \frac{\ln^m\left[1 - \dfrac{\ln(1-u)}{2\pi i}\right]}{1 - \dfrac{\ln(1-u)}{2\pi i}} - \frac{\ln^m\left[1 + \dfrac{\ln(1-u)}{2\pi i}\right]}{1 + \dfrac{\ln(1-u)}{2\pi i}} \right\} = \tag{18}$$

$$= \frac{(-1)^m m!}{\pi} \sum_{n=1}^{\infty} \frac{u^n}{n!} \cdot \sum_{k=0}^{\lfloor \frac{1}{2} n \rfloor} \frac{(-1)^k \big|S_1(2k+2, m+1)\big| \cdot \big|S_1(n, 2k+1)\big|}{(2\pi)^{2k+1}}$$

which uniformly holds in $|u| < 1$ and also is valid for $u = 1$.[15] Substituting (18) into (12) and performing the term-by-term integration from $u = 0$ to $u = 1$ yields the following series representation for mth generalized Euler's constant

$$\gamma_m = \frac{1}{2}\delta_{m,0} + \frac{(-1)^m m!}{\pi} \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} \sum_{k=0}^{\lfloor \frac{1}{2} n \rfloor} \frac{(-1)^k \big|S_1(2k+2, m+1)\big| \cdot \big|S_1(n, 2k+1)\big|}{(2\pi)^{2k+1}} \tag{19}$$

where $m = 0, 1, 2, \ldots$. In particular, for Euler's constant and first Stieltjes constant, we have following series expansions

$$\begin{aligned}
\gamma &= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} \sum_{k=0}^{\lfloor \frac{1}{2} n \rfloor} \frac{(-1)^k \cdot (2k+1)! \cdot \big|S_1(n, 2k+1)\big|}{(2\pi)^{2k+1}} \\
&= \frac{1}{2} + \frac{1}{2\pi^2} + \frac{1}{8\pi^2} + \frac{1}{18}\left(\frac{1}{\pi^2} - \frac{3}{4\pi^4}\right) + \frac{3}{96}\left(\frac{1}{\pi^2} - \frac{3}{2\pi^4}\right) \\
&\quad + \frac{1}{600}\left(\frac{12}{\pi^2} - \frac{105}{4\pi^4} + \frac{15}{4\pi^6}\right) + \frac{1}{4320}\left(\frac{60}{\pi^2} - \frac{675}{4\pi^4} + \frac{225}{4\pi^6}\right) + \ldots
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} \sum_{k=0}^{\lfloor \frac{1}{2} n \rfloor} \frac{(-1)^k \cdot (2k+1)! \cdot H_{2k+1} \cdot \big|S_1(n, 2k+1)\big|}{(2\pi)^{2k+1}} \\
&= -\frac{1}{2\pi^2} - \frac{1}{8\pi^2} - \frac{1}{18}\left(\frac{1}{\pi^2} - \frac{11}{8\pi^4}\right) - \frac{3}{96}\left(\frac{1}{\pi^2} - \frac{11}{4\pi^4}\right) \\
&\quad - \frac{1}{600}\left(\frac{12}{\pi^2} - \frac{385}{8\pi^4} + \frac{137}{16\pi^6}\right) - \frac{1}{4320}\left(\frac{60}{\pi^2} - \frac{2475}{8\pi^4} + \frac{2055}{16\pi^6}\right) - \ldots
\end{aligned} \tag{20}$$

respectively. As one can easily notice, each coefficient of these expansions contains polynomials in π^{-2} with rational coefficients. The rate of convergence of this series, depicted in Fig. 1, is relatively slow and depends, at least for the moderate number of terms, on m: the greater the order m, the slower the convergence. A more accurate description of this dependence, as well as the exact value of the rate of convergence, both require a detailed convergence analysis of (19), which is performed in the next section.

[15] The unit radius of convergence of this series is conditioned by the singularity the most closest to the origin. Such singularity is a branch point located at $u = 1$. Note also that since the series is convergent for $u = 1$ as well, in virtue of Abel's theorem on power series, it is uniformly convergent everywhere on the disc $|u| \leqslant 1 - \varepsilon$, where positive parameter ε can be made as small as we please.

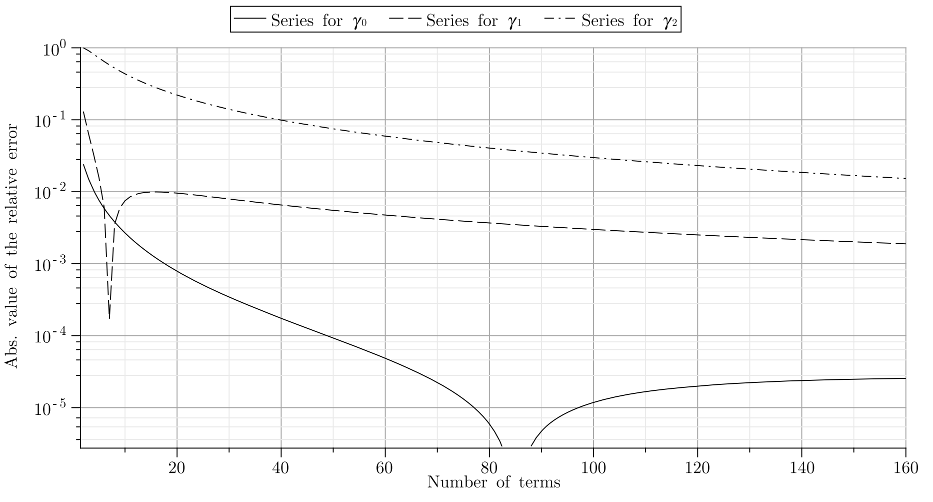


Fig. 1. Absolute values of relative errors of the series expansion for γ_0, γ_1 and γ_2 given by (19)–(20), logarithmic scale.

2.2. Convergence analysis of the derived series

The convergence analysis of series (19) consists in the study of its general term, which is given by the finite truncated sum over index k. This sum has only odd terms, and hence, by elementary transformations, may be reduced to that containing both odd and even terms

$$\sum_{k=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^k\,\frac{\big|S_1(2k+2,m+1)\big|\cdot\big|S_1(n,2k+1)\big|}{(2\pi)^{2k+1}} =$$
$$= \sum_{k=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^{\frac{1}{2}(2k+1)-\frac{1}{2}}\,\frac{\big|S_1(2k+1+1,m+1)\big|\cdot\big|S_1(n,2k+1)\big|}{(2\pi)^{2k+1}}$$
$$= \frac{1}{2}\sum_{l=1}^{n}\big[1-(-1)^l\big]\cdot(-1)^{\frac{1}{2}(l-1)}\cdot\frac{\big|S_1(l+1,m+1)\big|\cdot\big|S_1(n,l)\big|}{(2\pi)^l} = \ldots \tag{21}$$

where, in the last sum, we changed the summation index by putting $l = 2k + 1$. Now, from the second integral formula for the unsigned Stirling numbers of the first kind, see (A.4), it follows that

$$(-1)^{\frac{1}{2}(l-1)}\cdot\frac{\big|S_1(l+1,m+1)\big|}{(2\pi)^l} = \frac{(-1)^m}{2\pi}\cdot\frac{(l+1)!}{(m+1)!}\oint\limits_{|z|=r}\left[+\frac{i}{2\pi z}\right]^l\frac{\ln^{m+1}(1-z)}{z^2}\,dz$$

$$(-1)^l\cdot(-1)^{\frac{1}{2}(l-1)}\cdot\frac{\big|S_1(l+1,m+1)\big|}{(2\pi)^l} = \frac{(-1)^m}{2\pi}\cdot\frac{(l+1)!}{(m+1)!}\oint\limits_{|z|=r}\left[-\frac{i}{2\pi z}\right]^l\frac{\ln^{m+1}(1-z)}{z^2}\,dz$$

where $0 < r < 1$. Therefore, since $(l+1)! = \int x^{l+1} e^{-x} dx$ taken from 0 to ∞, the last sum in (21) reduces to the following integral representation

$$\begin{aligned}
\ldots &= \frac{(-1)^m}{4\pi(m+1)!} \sum_{l=1}^{n} \left|S_1(n,l)\right| \cdot (l+1)! \cdot \oint\limits_{|z|=r} \left[\left(\frac{i}{2\pi z}\right)^{l} - \left(-\frac{i}{2\pi z}\right)^{l}\right] \frac{\ln^{m+1}(1-z)}{z^2}\, dz \\
&= \frac{(-1)^m}{4\pi(m+1)!} \times \\
&\times \int\limits_0^\infty \left[\sum_{l=1}^{n} \left|S_1(n,l)\right| \oint\limits_{|z|=r} \left[\left(\frac{ix}{2\pi z}\right)^{l} - \left(-\frac{ix}{2\pi z}\right)^{l}\right] \frac{\ln^{m+1}(1-z)}{z^2}\, dz\right] x\, e^{-x}\, dx \\
&= \frac{(-1)^m}{4\pi(m+1)!} \cdot \oint\limits_{|z|=r} \frac{\ln^{m+1}(1-z)}{z^2} \left\{\int\limits_0^\infty \left[\left(\frac{ix}{2\pi z}\right)_n - \left(-\frac{ix}{2\pi z}\right)_n\right] x\, e^{-x}\, dx\right\} dz
\end{aligned} \tag{22}$$

The integral in curly brackets is difficult to evaluate in a closed-form, but at large n, its asymptotical value may be readily obtained. Function $1/\Gamma(z)$ is analytic on the entire complex z-plane, and hence, can be expanded into the MacLaurin series

$$\frac{1}{\Gamma(z)} = z + \gamma z^2 + \left(\frac{\gamma^2}{2} - \frac{\pi^2}{12}\right) z^3 + \ldots \equiv \sum_{k=1}^{\infty} z^k a_k\,, \qquad |z| < \infty\,, \tag{23}$$

where

$$a_k \equiv \frac{1}{k!} \cdot \left[\frac{1}{\Gamma(z)}\right]^{(k)}_{z=0} = \frac{(-1)^k}{\pi\, k!} \cdot \Big[\sin \pi x \cdot \Gamma(x)\Big]^{(k)}_{x=1}$$

see e.g. [3, p. 256, nº 6.1.34], [188, pp. 344 & 349], [77]. Using Stirling's approximation for the Pochhammer symbol (6), we have for sufficiently large n

$$\left(\frac{ix}{2\pi z}\right)_n - \left(-\frac{ix}{2\pi z}\right)_n \sim \frac{n^{\frac{ix}{2\pi z}} \cdot \Gamma(n)}{\Gamma\left(\frac{ix}{2\pi z}\right)} - \frac{n^{-\frac{ix}{2\pi z}} \cdot \Gamma(n)}{\Gamma\left(-\frac{ix}{2\pi z}\right)} = \tag{24}$$

$$= (n-1)! \left[\exp\left(\frac{ix\ln n}{2\pi z}\right) \sum_{k=1}^{\infty} a_k \left(\frac{i\,x}{2\pi z}\right)^{k} - \exp\left(-\frac{ix\ln n}{2\pi z}\right) \sum_{k=1}^{\infty} (-1)^k a_k \left(\frac{i\,x}{2\pi z}\right)^{k}\right]$$

Substituting this approximation into the integral in curly brackets from (22), performing the term-by-term integration[16] and taking into account that $z^{-s}\Gamma(s) = \int x^{s-1} e^{-zx} dx$ taken over $x \in [0, \infty)$, yields

[16] Series (23)–(24) being uniformly convergent.

$$\int_0^\infty \left[\left(\frac{ix}{2\pi z}\right)_n - \left(-\frac{ix}{2\pi z}\right)_n\right] x\, e^{-x}\, dx \sim$$

$$\sim (n-1)! \sum_{k=1}^{\infty} a_k \left(\frac{i}{2\pi z}\right)^k \cdot (k+1)! \left\{\left[1+\frac{i\ln n}{2\pi z}\right]^{-k-2} - (-1)^k \left[1-\frac{i\ln n}{2\pi z}\right]^{-k-2}\right\}$$

$$\sim (n-1)! \cdot \frac{32\, i\pi^3 z^3 \bigl(4\pi^2 z^2 - 3\ln^2 n\bigr)}{\bigl(4\pi^2 z^2 + \ln^2 n\bigr)^3}\,, \qquad n\to\infty\,, \tag{25}$$

where, at the final stage, we retained only the first significant term corresponding to factor $k = 1$.[17] Now, if $|z| \leqslant 1-e^{-1} \approx 0.63$, then the principal branch of $\bigl|\ln^{m+1}(1-z)\bigr| \leqslant 1$ independently of m and $\arg z$. Analogously, one can always find such sufficiently large n_0, that for any however small $\varepsilon > 0$,

$$\left|\frac{32\, i\pi^3 z^3 \bigl(4\pi^2 z^2 - 3\ln^2 n\bigr)}{\bigl(4\pi^2 z^2 + \ln^2 n\bigr)^3}\right| < \varepsilon\,, \qquad n \geqslant n_0\,, \tag{26}$$

on the circle $|z| = 1-e^{-1}$ (for example, if $\varepsilon = 1$, then $n_0 = 1222$; if $\varepsilon = 0.1$, then $n_0 = 38\,597$; if $\varepsilon = 0.01$, then $n_0 = 33\,220\,487$; *etc.*).[18] Combining all these results and taking into account that $|dz| = |z|\, d\arg z$, we conclude that

$$\frac{1}{n\cdot n!}\left|\sum_{k=0}^{\lfloor \frac{1}{2}n\rfloor} \frac{(-1)^k \bigl|S_1(2k+2, m+1)\bigr| \cdot \bigl|S_1(n, 2k+1)\bigr|}{(2\pi)^{2k+1}}\right| <$$

$$< \frac{1}{n^2}\cdot\frac{\varepsilon}{2\bigl(1-e^{-1}\bigr)(m+1)!} < \frac{C}{n^2}\,, \qquad n \geqslant n_0\,. \tag{27}$$

Numerical simulations, see Fig. 2, show that this simple inequality, valid for all m, may provide more or less accurate approximation for the general term of (22), and this greatly

[17] The second term of this sum, corresponding to $k = 2$, is

$$(n-1)! \cdot \frac{396\, i\gamma\pi^3 z^3 \bigl(4\pi^2 z^2 - \ln^2 n\bigr)\ln n}{\bigl(4\pi^2 z^2 + \ln^2 n\bigr)^4} = (n-1)! \cdot O\!\left(\frac{1}{\ln^5 n}\right), \qquad n\to\infty\,,$$

and hence, may be neglected at large n.

[18] Note that for fixed n, the left-hand side of (26) reaches its maximum when z is imaginary pure.

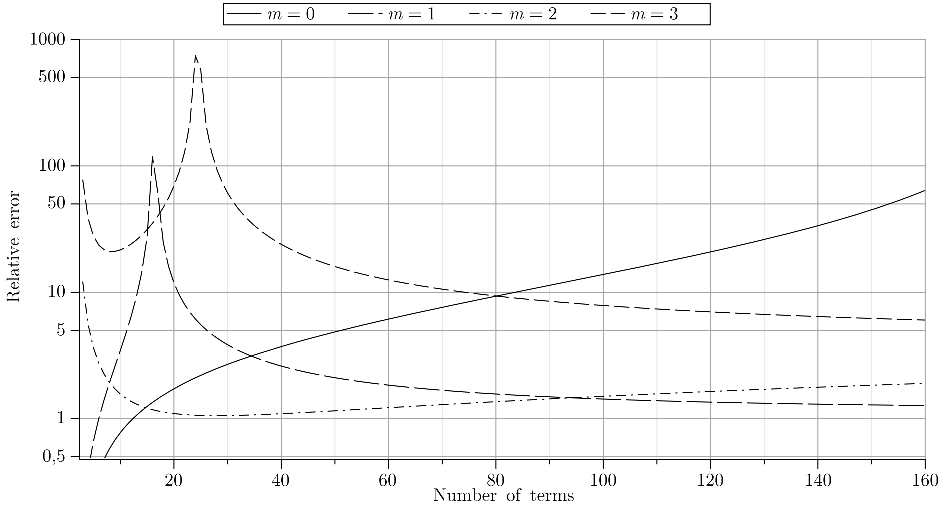


Fig. 2. Relative error between the upper bound and the left-hand side in (27) as a function of n for four different orders m, logarithmic scale (the curve in long dashes correspond to $m = 1$, that in short ones to $m = 3$). Results displayed above correspond to $C = 1/(2\pi)$.

depends on m. Moreover, the joint analysis of Figs. 1 and 2 also indicates that first partial sums of series (19) may behave quite irregularly. One of the reasons of such a behaviour is that for $1 \leqslant n \leqslant 53$, absolute value (26) increases, and it starts to decrease only after $n = 54$.[19] Notwithstanding, inequality (27) guarantees that in all cases, the discovered series for γ_m given by (19) converges for large n not worse than Euler's series $\sum n^{-2}$.

Asymptotics (25), as well as the rates of convergence previously obtained in [18], both suggest that the exact rate of convergence of series (19) may also involve logarithms. For instance, from [18, Sect. 3], it straightforwardly follows that the rate of convergence of this series at $m = 0$ is equal to $\sum (n \ln n)^{-2}$, see also footnote 4. Indeed, if we replace the integral in curly brackets from (22) by its first-order approximation (25), and then, evaluate the sum of corresponding residues at $z_{1,2} \equiv \pm \frac{i \ln n}{2\pi}$

$$\sum_{l=1}^{2} \operatorname*{res}_{z=z_l} \frac{z\big(4\pi^2 z^2 - 3\ln^2 n\big)\ln(1-z)}{\big(4\pi^2 z^2 + \ln^2 n\big)^3} = \frac{\ln^2 n - 4\pi^2}{8\pi^2\big(4\pi^2 + \ln^2 n\big)^2} \sim \frac{1}{8\pi^2 \ln^2 n}$$

$$\sum_{l=1}^{2} \operatorname*{res}_{z=z_l} \frac{z\big(4\pi^2 z^2 - 3\ln^2 n\big)\ln^2(1-z)}{\big(4\pi^2 z^2 + \ln^2 n\big)^3} = \frac{\ln^2 n \cdot \ln\big(4\pi^2 + \ln^2 n\big) + \ldots}{8\pi^2\big(4\pi^2 + \ln^2 n\big)^2} \sim \frac{2 \cdot \ln\ln n}{8\pi^2 \ln^2 n}$$

[19] On the circle $|z| = 1 - e^{-1}$, absolute value (26) has one of its third-order poles at $n = e^{2\pi(1-e^{-1})} \approx 53.08$. Other poles are located either below $n = 1$, e.g. $n = e^{-2\pi(1-e^{-1})} \approx 0.02$, or are complex. More precisely, all poles of this expression occur at $n = \big[\cos\big(2\pi(1-e^{-1})\cos\varphi\big) \mp i\sin\big(2\pi(1-e^{-1})\cos\varphi\big)\big]e^{\pm 2\pi(1-e^{-1})\sin\varphi}$, where $\varphi \equiv \arg z$.

$$\sum_{l=1}^{2} \operatorname*{res}_{z=z_l} \frac{z\bigl(4\pi^2 z^2 - 3\ln^2 n\bigr)\ln^3(1-z)}{\bigl(4\pi^2 z^2 + \ln^2 n\bigr)^3} =$$

$$= \frac{3\ln^2 n \cdot \bigl[\ln^2(2\pi + i\ln n) + \ln^2(2\pi - i\ln n)\bigr] + \ldots}{16\pi^2\bigl(4\pi^2 + \ln^2 n\bigr)^2} \sim \frac{3\cdot \ln^2 \ln n}{8\pi^2 \ln^2 n}$$

and so on, we find that

$$\frac{1}{n\cdot n!}\sum_{k=0}^{\lfloor\frac{1}{2}n\rfloor}(-1)^k \frac{\bigl|S_1(2k+2,m+1)\bigr|\cdot\bigl|S_1(n,2k+1)\bigr|}{(2\pi)^{2k+1}} \sim \tag{28}$$

$$\sim \frac{8\,i\,\pi^2\,(-1)^m}{n^2(m+1)!}\oint\limits_{|z|=r} \frac{z\bigl(4\pi^2 z^2 - 3\ln^2 n\bigr)\ln^{m+1}(1-z)}{\bigl(4\pi^2 z^2 + \ln^2 n\bigr)^3}\,dz \sim (-1)^{m+1}\frac{2\pi}{m!}\cdot\frac{\ln^m \ln n}{n^2\ln^2 n}$$

in virtue of the Cauchy residue theorem. Of course, this formula is only a rough approximation, because poles $z_{1,2}$ belongs to the disc $|z| = r < 1$ only if $1 \leqslant n \leqslant 535$, while formula (25) is a double first-order approximation and is accurate only for large n. Furthermore, residues were also evaluated only in the first approximation. However, obtained expression gives an idea of what the true rate of convergence of series (19) could be, and it also explains quite well why series for higher generalized Euler's constants converge more slowly than those for lower generalized Euler's constants. Moreover, this approximation agrees with the fact that (19) converges not worse than Euler's series, and also is consistent with the previously derived rate of convergence for γ from [18], which was obtained by another method.

3. Expansion of generalized Euler's constants γ_m into the formal series with rational coefficients

3.1. Introduction

Expansions into the series with rational coefficients is an interesting and challenging subject. There exist many such representations for Euler's constant γ and first of them date back to the XVIIIth century. For instance, from the Stirling series for the digamma function at $z = 1$, it straightforwardly follows that

$$\begin{aligned}\gamma &= \frac{1}{2} + \sum_{k=1}^{N}\frac{B_{2k}}{2k} + \theta\cdot\frac{B_{2N+2}}{2(N+1)}\\ &= \frac{1}{2} + \frac{1}{12} - \frac{1}{120} + \frac{1}{252} - \frac{1}{240} + \frac{1}{132} - \frac{691}{32\,760} + \ldots\end{aligned} \tag{29}$$

where $0 < \theta < 1$ and $N < \infty$.[20] A more general representation of the same kind may be obtained by Euler–MacLaurin summation

$$\gamma = \sum_{k=1}^{n}\frac{1}{k} - \ln n - \frac{1}{2n} + \sum_{k=1}^{N}\frac{B_{2k}}{2k\cdot n^{2k}} + \theta\cdot\frac{B_{2N+2}}{2(N+1)\cdot n^{2N+2}}\,, \tag{30}$$

where $0 < \theta < 1$, $N < \infty$ and n is positive integer, see e.g. [71, n° 377]. Two above series are *semi-convergent* (or *divergent enveloping*), i.e. they diverge as $N \to \infty$. The first known convergent series representation for Euler's constant with only rational terms, as far as we know, dates back to 1790 and is due to Gregorio Fontana and Lorenzo Mascheroni

$$\gamma = \sum_{n=1}^{\infty}\frac{|G_n|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \ldots \tag{31}$$

where rational coefficients G_n, known as *Gregory's coefficients*,[21] are given either via their generating function

$$\frac{z}{\ln(1+z)} = 1 + \sum_{n=1}^{\infty} G_n z^n, \qquad |z| < 1\,, \tag{32}$$

or explicitly

$$G_n = \frac{1}{n!}\sum_{l=1}^{n}\frac{S_1(n,l)}{l+1} = \frac{1}{n!}\int_0^1 (x-n+1)_n\,dx = -\frac{B_n^{(n-1)}}{(n-1)\,n!} = \frac{C_{1,n}}{n!} \tag{33}$$

[20] This result should be attributed to both Stirling and De Moivre, who originated Stirling series, see [172, p. 135] and [48] respectively (for more information on Stirling series, see also [54, part II, Chapter VI, p. 466], [58, p. 33], [25, p. 329], [135, p. 111], [184, §12-33], [86, §15-05], [99, p. 530], [46, p. 1], [107], [138, §4.1, pp. 293–294], [59, pp. 286–288], [3, n° 6.1.40–n° 6.1.41], [127]). Curiously, Srivastava and Choi [166, p. 6], did not notice the trivial connection between this series and the Stirling series for the digamma function and erroneously credited this result to Konrad Knopp, in whose book [99] it appears, with a slightly different remainder, on p. 527 (Knopp himself never claimed the authorship of this formula).

[21] These coefficients are also called *(reciprocal) logarithmic numbers*, *Bernoulli numbers of the second kind*, normalized *generalized Bernoulli numbers* $B_n^{(n-1)}$, *Cauchy numbers* and normalized *Cauchy numbers of the first kind* $C_{1,n}$. They were introduced by James Gregory in 1670 in the context of area's interpolation formula (which is known nowadays as *Gregory's interpolation formula*) and were subsequently rediscovered in various contexts by many famous mathematicians, including Gregorio Fontana, Lorenzo Mascheroni, Pierre–Simon Laplace, Augustin–Louis Cauchy, Jacques Binet, Ernst Schröder, Oskar Schlömilch, Charles Hermite, Jan C. Kluyver and Joseph Ser [146, vol. II, pp. 208–209], [178, vol. 1, p. 46, letter written on November 23, 1670 to John Collins], [86, pp. 266–267, 284], [64, pp. 75–78], [35, pp. 395–396], [119, pp. 21–23], [111, T. IV, pp. 205–207], [21, pp. 53–55], [181], [64, pp. 192–194], [115,183,161,160], [78, pp. 65, 69], [95,162]. For more information about these important coefficients, see [135, pp. 240–251], [136], [90, p. 132, Eq. (6), p. 138], [91, p. 258, Eq. (14)], [92, pp. 266–267, 277–280], [133,134,170], [171, pp. 106–107], [47], [185, p. 190], [71, p. 45, n° 370], [13, vol. III, pp. 257–259], [167], [109, p. 229], [142, n° 600, p. 87], [99, p. 216, n° 75-a] [40, pp. 293–294, n° 13], [30,82,191,6,192,28], [124, Eq. (3)], [123,129], [9, pp. 128–129], [11, Chapt. 4], [105,18].

This series was first studied by Fontana, who, however, failed to find a constant to which it converges. Mascheroni identified this *Fontana's constant* and showed that it equals Euler's constant [119, pp. 21–23]. This series was subsequently rediscovered many times, in particular, by Ernst Schröder in 1879 [161, p. 115, Eq. (25a)], by Niels E. Nørlund in 1923 [135, p. 244], by Jan C. Kluyver in 1924 [95], by Charles Jordan in 1929 [90, p. 148], by Kenter in 1999 [93], by Victor Kowalenko in 2008 [104,103]. An expansion of a similar nature

$$\gamma = 1 - \sum_{n=1}^{\infty} \frac{C_{2,n}}{n \cdot (n+1)!} = 1 - \frac{1}{4} - \frac{5}{72} - \frac{1}{32} - \frac{251}{14\,400} - \frac{19}{1728} - \frac{19\,087}{2\,540\,160} - \ldots \tag{34}$$

where rational numbers $C_{2,n}$, known as *Cauchy numbers of the second kind*[22]

$$\begin{cases} \dfrac{z}{(1+z)\ln(1+z)} = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n z^n C_{2,n}}{n!} \\[2ex] C_{2,n} = \displaystyle\sum_{l=1}^{n} \frac{|S_1(n,l)|}{l+1} = \int_0^1 (x)_n \, dx = |B_n^{(n)}| \end{cases} \tag{35}$$

follows from a little-known series for the digamma function given by Jacques Binet in 1839 [17, p. 257, Eq. (81)] and rediscovered later by Niels E. Nørlund in his monograph [135, p. 244].[23] Series

$$\gamma = 1 - \sum_{n=1}^{\infty} \sum_{k=2^{n-1}}^{2^n-1} \frac{n}{(2k+1)(2k+2)} = 1 - \sum_{n=1}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} \frac{(-1)^{k+1} n}{k} = 1 - \frac{1}{12} - \frac{43}{420}$$
$$- \frac{20\,431}{240\,240} - \frac{2\,150\,797\,323\,119}{36\,100\,888\,223\,400} - \frac{9\,020\,112\,358\,835\,722\,225\,404\,403}{236\,453\,376\,820\,564\,453\,502\,272\,320} - \ldots \tag{36}$$

was given in the first form by Niels Nielsen in 1897 [131, Eq. (6)], and in the second form by Ernst Jacobsthal in 1906 [85, Eqs. (8)]. The same series (in various forms) was independently obtained by Addison in 1967 [5] and by Gerst in 1969 [60]. The famous series

$$\gamma = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \lfloor \log_2 n \rfloor = \frac{1}{2} - \frac{1}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \ldots \tag{37}$$

[22] These numbers, called by some authors signless *generalized Bernoulli numbers* $|B_n^{(n)}|$ and signless *Nørlund numbers*, are much less famous than Gregory's coefficients G_n, but their study is also very interesting, see [135, pp. 150–151], [47, p. 12], [136], [13, vol. III, pp. 257–259], [40, pp. 293–294, n° 13], [81,6,192,143,18].

[23] Strictly speaking, Binet found only first four coefficients of the corresponding series for the digamma function and incorrectly calculated the last coefficient (for $K(5)$ he took $\frac{245}{3}$ instead of $\frac{245}{6}$ [17, p. 237]), but otherwise his method and derivations are correct. It is also notable that Binet related coefficients $K(n)$ to the Stirling numbers and provided two different ways for their computation, see [18, Final remark].

was first given by Ernst Jacobsthal in 1906 [85, Eqs. (9)] and subsequently rediscovered by many writers, including Giovanni Vacca [180], H.F. Sandham [153], D.F. Barrow, M.S. Klamkin, N. Miller [12] and Gerst [60].[24] Series

$$\gamma=\sum_{n=m}^{\infty}\frac{\beta_n}{n}\lfloor\log_m n\rfloor,\qquad \beta_n=\begin{cases}m-1, & n=\text{multiple of } m\\ -1, & n\neq\text{multiple of } m\end{cases}\tag{38}$$

which generalizes foregoing Jacobsthal–Vacca's series (37), is due to Jan C. Kluyver who discovered it in 1924 [95].

In contrast, as concerns generalized Euler's constants γ_m the results are much more modest. In 1912 Hardy [73], by trying to generalize Jacobsthal–Vacca's series (37) to first generalized Euler's constant, obtained the following series

$$\gamma_1=\frac{\ln 2}{2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\lfloor\log_2 n\rfloor\cdot\big(2\log_2 n-\lfloor\log_2 2n\rfloor\big)\tag{39}$$

which is, however, not a full generalization of Jacobsthal–Vacca's series since it also contains irrational coefficients. In 1924–1927, Kluyver tried, on several occasions [94,96], to better Hardy's result and to obtain series for γ_m with rational terms only, but these attempts were not successful. There also exist formulæ similar to Hardy's series. For instance,

$$\sum_{n=1}^{\infty}\frac{H_n^m-(\gamma+\ln n)^m}{n}=\begin{cases}-\gamma_1-\frac{1}{2}\gamma^2+\frac{1}{2}\zeta(2)\,, & m=1\\ -\gamma_2-\frac{2}{3}\gamma^3-2\gamma_1\gamma+\frac{5}{3}\zeta(3)\,, & m=2\\ -\gamma_3-\frac{3}{4}\gamma^4-3\gamma_2\gamma-3\gamma_1\gamma^2+\frac{43}{8}\zeta(4)\,, & m=3\end{cases}$$

see e.g. [57,120].[25] Besides, several asymptotical representations similar to Hardy's formula are also known for γ_m. For instance, Nikolai M. Günther and Rodion O. Kuzmin gave the following formula

$$\gamma_1=\sum_{k=1}^{n}\frac{\ln k}{k}-\frac{\ln^2 n}{2}-\frac{\ln n}{2\,n}-\frac{1-\ln n}{12\,n^2}+\theta\cdot\frac{11-6\ln n}{720\,n^4}\tag{40}$$

where $0<\theta<1$, see [71, n° 388].[26] M.I. Israilov [84] generalized expression (40) and showed that the mth Stieltjes constant may be given by a similar semi-convergent asymptotical series

[24] It should be remarked that this series is often incorrectly attributed to Vacca, who only rediscovered it. This error initially is due to Glaisher, Hardy and Kluyver, see e.g. their works [63,73,95]. It was only much later that Stefan Krämer [105] correctly attributed this series to Jacobsthal [85].

[25] Cases $m=1$ and $m=2$ are discussed in cited references. Formula for $m=3$ was kindly communicated to the author by Roberto Tauraso.

[26] In the third edition of [71], published in 1947, there are two errors in exercise n° 388: the value of γ_1 is given as $-0.073927\ldots$ instead of $-0.072815\ldots$, and the denominator of the last term has the value 176 instead of 720. These errors were corrected in the fourth edition of this book, published in 1951.

$$\gamma_m = \sum_{k=1}^{n} \frac{\ln^m k}{k} - \frac{\ln^{m+1} n}{m+1} - \frac{\ln^m n}{2n} - \\ - \sum_{k=1}^{N-1} \frac{B_{2k}}{(2k)!} \left[\frac{\ln^m x}{x}\right]^{(2k-1)}_{x=n} - \theta \cdot \frac{B_{2N}}{(2N)!} \left[\frac{\ln^m x}{x}\right]^{(2N-1)}_{x=n} \tag{41}$$

where $m = 0, 1, 2, \ldots$, $0 < \theta < 1$, and integers n and N may be arbitrary chosen provided that N remains finite.[27,28] Using various series representations for the ζ-function, it is also possible to obtain corresponding series for the Stieltjes constants. For instance, from Ser's series for the ζ-function (see footnote 30), it follows that

$$\gamma_m = -\frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\ln^{m+1}(k+1)}{k+1}, \qquad m = 0, 1, 2 \ldots \tag{42}$$

An equivalent result was given by Donal F. Connon [41][29]

$$\gamma_m = -\frac{1}{m+1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln^{m+1}(k+1), \qquad m = 0, 1, 2 \ldots \tag{43}$$

who used Hasse's series for the ζ-function.[30] Similarly, using another Ser's series expansions for $\zeta(s)$,[31] we conclude that

[27] Note that at fixed N, the greater the number n, the more accurate this formula; at $n \to \infty$ it straightforwardly reduces to (3). It seems also appropriate to note here that, although Günther, Kuzmin and Israilov obtained (40) and (41) independently, both these formulæ may be readily derived from an old semi-convergent series for the ζ-function, given, for example, by Jørgen P. Gram in 1895 [69, p. 304, 2nd unnumbered formula] (this series for $\zeta(s)$ may be, of course, much older since it is a simple application of the Euler–Maclaurin summation formula; note also that Gram uses a slightly different convention for the Bernoulli numbers).

[28] In [84, Eq. (3)], there is a misprint: in the denominator of the second sum $2r$ should be replaced by $(2r)!$ [this formula appears correctly on p. 101 [84], but with a misprint in Eq. (3) on p. 98]. This misprint was later reproduced in [151, Theorem 0.3].

[29] Strictly speaking, Connon [41, p. 1] gave this formula for the generalized Stieltjes constants $\gamma_m(v)$, of which ordinary Stieltjes constants are simple particular cases $\gamma_m = \gamma_m(1)$, see e.g. [19, Eqs. (1)–(2)].

[30] The series representation for $\zeta(s)$, which is usually attributed to Helmut Hasse, is actually due to Joseph Ser who derived it in 1926 in a slightly different form. The series in question is

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+2} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s} = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{1-s}$$

The first variant was given by Ser in 1926 in [162, p. 1076, Eq. (7)], the second variant was given by Hasse in 1930 [75, pp. 460–461]. The equivalence between two forms follows from the recurrence relation for the binomial coefficients. It is interesting that many writers do not realize that Ser's formula and Hasse's formula are actually the same (the problem is also that Ser's paper [162] contains errors, e.g. in Eq. (2), p. 1075, the last term in the second line should be $(-1)^n (n+1)^{-s}$ instead of $(-1)^n n^{-s}$). An equivalent series representation was also independently discovered by Jonathan Sondow in 1994 [165].

[31] Ser's formula [162, p. 1076, Eq. (4)], corrected (see footnote 30) and written in our notations, reads

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s}.$$

$$\gamma_m = \sum_{n=0}^{\infty} \big|G_{n+1}\big| \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\ln^m(k+1)}{k+1}\,, \qquad m = 0, 1, 2\ldots \tag{44}$$

where G_n are Gregory's coefficients, see (31)–(33).[32] This series was also independently discovered by Marc-Antoine Coppo in 1997 [45, p. 355, Eq. (5)] by the method of finite differences. Several more complicated series representations for γ_m with irrational coefficients may be found in [176,113,84,168,128,186,38,39].[33]

3.2. Derivation of the series expansion

Consider now series (19). A formal rearrangement of this expression may produce a series with rational terms only for γ_m. In view of the fact that

$$\zeta(k+1) = \sum_{n=k}^{\infty} \frac{\big|S_1(n,k)\big|}{n\cdot n!} = \sum_{n=1}^{\infty} \frac{\big|S_1(n,k)\big|}{n\cdot n!}\,, \qquad k = 1, 2, 3, \ldots \tag{45}$$

see e.g. [92, pp. 166, 194–195],[34] and that

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}\cdot B_{2k}}{2\cdot(2k)!} = \frac{(2\pi)^{2k}\cdot |B_{2k}|}{2\cdot(2k)!}\,, \qquad k = 1, 2, 3, \ldots$$

the formal interchanging of the order of summation in (19) leads to

$$\frac{(-1)^m m!}{\pi} \sum_{n=1}^{\infty} \frac{1}{n\cdot n!} \sum_{k=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^k \big|S_1(2k+2, m+1)\big| \cdot \big|S_1(n, 2k+1)\big|}{(2\pi)^{2k+1}} \asymp$$

---

[32] Note that for $m = 0$, the latter series reduces to Fontana–Mascheroni's series (31).

[33] Apart from formula (41), in [84], Israilov also obtained several other series representations for γ_m. Stankus [168] showed that first two Stieltjes constants may be represented by the series involving the divisor function. Works of several authors showing that Stieltjes constants are related to series containing nontrivial zeros of the ζ-function are summarized in [186]. Coffey gave various series representations for γ_m in [38,39]. However, we also noted that [38,39] both contain numerous rediscoveries, as well as inaccuracies in attribution of formulæ (see also [43]). For instance, "Addison-type series" [39, Eq. (1.3)] are actually due to Nielsen and Jacobsthal, see our Eq. (36) above. Numbers p_{n+1} are simply signless Gregory's coefficients $|G_n|$ and their asymptotics, Eq. (4.8)/(2.82) [38, p. 473/31], is known at least since 1879 [161, Eqs. (25)–(25a)] (see also [170], [171, pp. 106–107]). Their "full" asymptotics, Eq. (4.10)/(2.84) [38, p. 473/31], is also known and was given, for example, by Van Veen in 1951 [181], [136]. Representation (2.17) [38, p. 455/13] is due to Hermite, see Eq. (13) [19, p. 541]. Formula (1.17) [39, p. 2052] was already obtained at least twice: by Binet in 1839 [17, p. 257] and by Nørlund in 1923 [135, p. 244], see details in [18, Final remark]. Formula (1.18) [39, p. 2052] straightforwardly follows from Ser's results [162] dating back to 1926, *etc.*

[34] See also [163,154], where this important result was rediscovered much later. By the way, this formula may be generalized to the Hurwitz ζ-function

$$\zeta(k+1, v) = \sum_{n=k}^{\infty} \frac{|S_1(n,k)|}{n\cdot (v)_n}\,, \qquad k = 1, 2, 3, \ldots\,, \qquad \operatorname{Re} v > 0\,,$$

see [18]. At $n\to\infty$, the general term of this series is $O\left(\dfrac{\ln^{k-1} n}{n^{v+1}}\right)$.

$$\asymp \frac{(-1)^m m!}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k\big|S_1(2k+2,m+1)\big|}{(2\pi)^{2k+1}}\underbrace{\sum_{n=1}^{\infty}\frac{\big|S_1(n,2k+1)\big|}{n\cdot n!}}_{\zeta(2k+2)} =$$

$$= (-1)^m m!\sum_{k=1}^{\infty}\frac{\big|S_1(2k,m+1)\big|\cdot B_{2k}}{(2k)!}$$

Remarking that such operations are not permitted when series are not absolutely convergent (that is why we wrote $\asymp$ instead of $=$), we understand why the resulting series diverges. However, since this series is alternating, for any prescribed m and N, one can always find such $\theta\in(0,1)$, generally depending on m and N, that

$$\gamma_m = \frac{1}{2}\delta_{m,0} + (-1)^m m!\cdot\sum_{k=1}^{N}\frac{\big|S_1(2k,m+1)\big|\cdot B_{2k}}{(2k)!} + \tag{46}$$

$$+\,\theta\cdot\frac{(-1)^m m!\cdot\big|S_1(2N+2,m+1)\big|\cdot B_{2N+2}}{(2N+2)!} =$$

$$=\begin{cases} \dfrac{1}{2}+\dfrac{1}{12}-\dfrac{1}{120}+\dfrac{1}{252}-\dfrac{1}{240}+\dfrac{1}{132}-\ldots & m=0\\[2mm] -\dfrac{1}{12}+\dfrac{11}{720}-\dfrac{137}{15\,120}+\dfrac{121}{11\,200}-\dfrac{7129}{332\,640}+\dfrac{57\,844\,301}{908\,107\,200}-\ldots & m=1\\[2mm] +0-\dfrac{1}{60}+\dfrac{5}{336}-\dfrac{469}{21\,600}+\dfrac{6515}{133\,056}-\dfrac{131\,672\,123}{825\,552\,000}+\dfrac{63\,427}{89\,100}-\ldots & m=2\\[2mm] -0+\dfrac{1}{120}-\dfrac{17}{1008}+\dfrac{967}{28\,800}-\dfrac{4523}{49\,896}+\dfrac{33\,735\,311}{101\,088\,000}-\dfrac{9\,301\,169}{5\,702\,400}+\ldots & m=3 \end{cases}$$

holds strictly.[35] Moreover, taking into account (16), above series may be always written in a form without Stirling numbers. For instance, for Euler's constant and for first three Stieltjes constants, it becomes

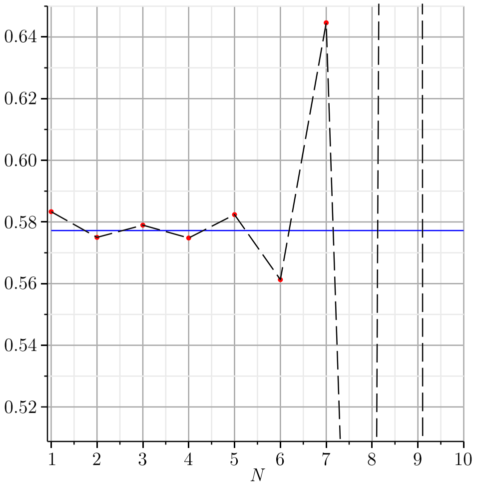
$$\gamma = +\frac{1}{2}\left\{1+\sum_{k=1}^{N}\frac{B_{2k}}{k}+\theta\cdot\frac{B_{2N+2}}{N+1}\right\}$$

$$\gamma_1 = -\frac{1}{2}\sum_{k=1}^{N}\frac{B_{2k}\cdot H_{2k-1}}{k}+\theta\cdot\frac{B_{2N+2}\cdot H_{2N+1}}{2N+2}$$

$$\gamma_2 = +\frac{1}{2}\sum_{k=1}^{N}\frac{B_{2k}\cdot\big\{H_{2k-1}^2-H_{2k-1}^{(2)}\big\}}{k}+\theta\cdot\frac{B_{2N+2}\cdot\big\{H_{2N+1}^2-H_{2N+1}^{(2)}\big\}}{2N+2}$$

$$\gamma_3 = -\frac{1}{2}\sum_{k=1}^{N}\frac{B_{2k}\cdot\big\{H_{2k-1}^3-3H_{2k-1}H_{2k-1}^{(2)}+2H_{2k-1}^{(3)}\big\}}{k}+$$

[35] There is another way to obtain (46). Consider first (41) at $n=1$, and then use (13) to show that $\left[\frac{d^n}{dx^n}\frac{\ln^m x}{x}\right]_{x=1} = m!\,S_1(n+1,m+1)$.



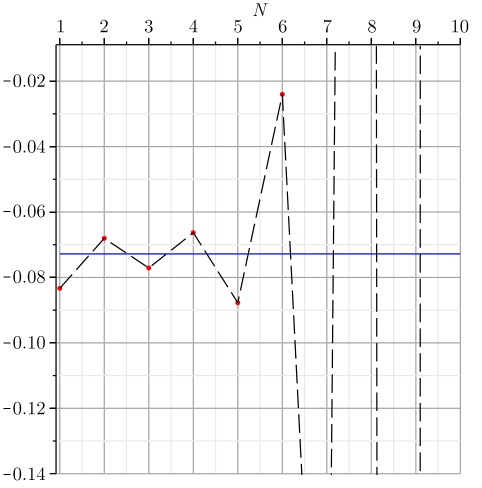


Fig. 3. Partial sums of series (47) for γ and γ_1 (on the left and on the right respectively) for $N = 1, 2, \ldots, 10$. Blue lines indicate the true value of γ and γ_1, while dashed black lines with the red points display corresponding partial sums given by (47). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$+ \theta \cdot \frac{B_{2N+2} \cdot \left\{ H^3_{2N+1} - 3H_{2N+1}H^{(2)}_{2N+1} + 2H^{(3)}_{2N+1} \right\}}{2N+2} \tag{47}$$

where $0 < \theta < 1$ and $N < \infty$ (these parameters are different in all equations, and in each equation θ, in general, depends on N). By the way, one may notice that first of these formulæ coincide with Stirling series (29), while other formulæ are, to our knowledge, new and seem to be never released before. Derived formal series are alternating and are sometimes referred to as *semi-convergent series* or *divergent enveloping series*.[36] It is also easy to see that they diverge very rapidly

$$\begin{aligned} \frac{\left|S_1(2k, m+1)\right| \cdot B_{2k}}{(2k)!} &\sim 2(-1)^{k-1} \frac{(2k-1)! \cdot \ln^m(2k-1)}{m! \cdot (2\pi)^{2k}} \\ &\sim \frac{2\sqrt{\pi}(-1)^{k-1}}{m!} \cdot \frac{\ln^m k}{\sqrt{k}} \cdot \left(\frac{k}{\pi e}\right)^{2k}, \end{aligned}$$

$k \to \infty$, $m = 0, 1, 2, \ldots$, so rapidly that even the corresponding power series $\sum \frac{|S_1(2k,m+1)|}{(2k)!} B_{2k} x^{2k}$ diverges everywhere.[37] Behaviour of this series for first two Stieltjes constants is shown in Fig. 3.

[36] These series were an object of study of almost all great mathematicians; the reader interested in a deeper study of these series may wish to consult the following literature: [22,51,74], [25, Chapt. XI], [141, Chapt. 4, §1], [99], exercises n° 374–n° 388 in [71, pp. 46–48], [187,139,182,46,50,138,20,144,145,53,118].

[37] Coefficients $\left|S_1(2k, m+1)\right|$ and Bernoulli numbers both grow very quickly: as $k \to \infty$ we have $\left|S_1(2k, m+1)\right| \sim (2k-1)! \ln^m(2k-1)/m!$, see (15), and $B_{2k} \sim 2(-1)^{k-1}(2\pi)^{-2k}(2k)!$, see e.g. [109, p. 5], [59, p. 261].

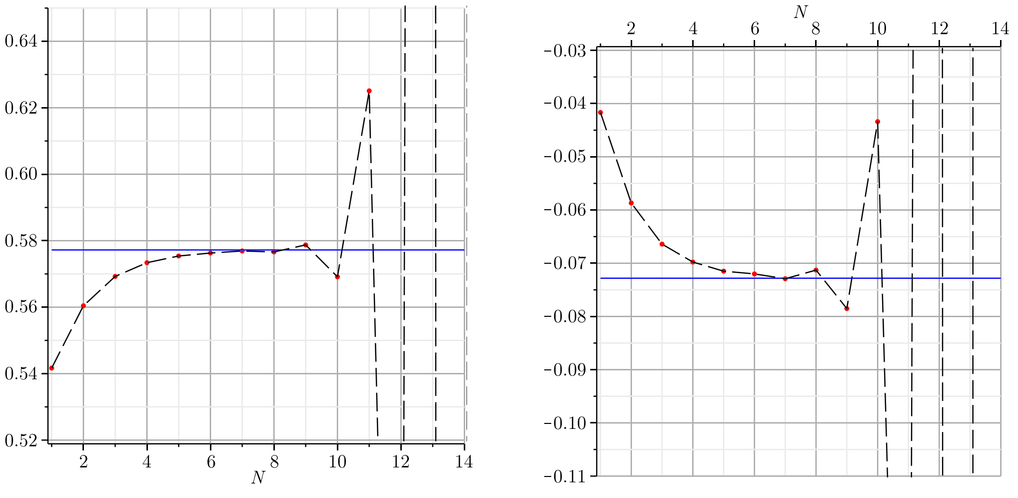


Fig. 4. Euler's transformation of series (47) for γ and γ_1 (on the left and on the right respectively) for $N = 1, 2, \ldots, 14$. Blue lines indicate the true value of γ and γ_1, while dashed black lines with the red points display corresponding partial sums given by (48). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3.3. Some series transformations applied to the derived divergent series

In order to convert (46) into a convergent series, one may try to apply various series transformations and regularization procedures. However, since (46) is strongly divergent, the use of standard summation methods may not result in a convergent series. For example, applying Euler's transformation[38] we obtain another series with rational coefficients only

$$\gamma_m = \frac{1}{2}\delta_{m,0} + (-1)^m m! \sum_{k=1}^{N} \frac{1}{2^k} \sum_{n=1}^{k} \frac{\bigl|S_1(2n, m+1)\bigr| \cdot B_{2n}}{(2n)!} \cdot \binom{k-1}{n-1} + R_m(N) =$$

$$= \begin{cases} \dfrac{1}{2} + \dfrac{1}{24} + \dfrac{3}{160} + \dfrac{89}{10\,080} + \dfrac{37}{8960} + \dfrac{299}{147\,840} + \ldots & m=0 \\[2ex] -\dfrac{1}{24} - \dfrac{49}{2880} - \dfrac{187}{24\,192} - \dfrac{5431}{1\,612\,800} - \dfrac{91\,151}{53\,222\,400} - \ldots & m=1 \\[2ex] -\dfrac{1}{240} - \dfrac{31}{13\,440} - \dfrac{4093}{2\,419\,200} - \dfrac{50\,789}{106\,444\,800} - \dfrac{602\,325\,403}{581\,188\,608\,000} + \ldots & m=2 \\[2ex] +\dfrac{1}{480} - \dfrac{1}{40\,320} + \dfrac{1609}{3\,225\,600} - \dfrac{120\,749}{159\,667\,200} + \dfrac{694\,773\,847}{498\,161\,664\,000} - \ldots & m=3 \end{cases} \tag{48}$$

[38] See e.g. [99, pp. 244–246], [100, pp. 144 & 170–171], [182, pp. 269–278 & 305–306].

which are all divergent (their remainder $R(N) \to \infty$ as $N \to \infty$). At the same time, these series behave much better than (46). Thus, the series for γ starts to clearly diverge only from $N \geqslant 10$, and that for γ_1 only from $N \geqslant 8$, see Fig. 4. The minimum error for the first series corresponds to $N = 7$ and equals 3×10^{-4}, that for γ_1 also corresponds to $N = 7$ and equals 9×10^{-5}. Attempts to regularize series (46)–(47) with the help of Cesàro summation are also fruitless since its general term grows more rapidly than k at $k \to \infty$. Similarly, Borel's summation does not provide a convergent result.

3.4. An estimate for generalized Euler's constants

Finally, we note that derived formal series (47) provides an estimation for generalized Euler's constants. Since this series is enveloping, its true value always lies between two neighbouring partial sums. If, for example, we retain only the first non-vanishing term, the mth Stieltjes constant will be stretched between the first and second non-vanishing partial sums. Thus, accounting for known properties of the Stirling numbers of the first kind

$$\begin{aligned}
\big|S_1(m+1,m+1)\big| &= 1 \\
\big|S_1(m+2,m+1)\big| &= \frac{1}{2}(m+1)(m+2) \\
\big|S_1(m+3,m+1)\big| &= \frac{1}{24}(m+1)(m+2)(m+3)(3m+8) \\
\big|S_1(m+4,m+1)\big| &= \frac{1}{48}(m+1)(m+2)(m+3)^2(m+4)^2
\end{aligned}$$

which are valid for $m = 0, 1, 2, \ldots$, we have

$$\begin{gathered}
(-1)^{\frac{1}{2}(m+1)} \frac{\big|B_{m+1}\big|}{m+1} < \gamma_m < (-1)^{\frac{1}{2}(m+1)} \left\{ \frac{\big|B_{m+1}\big|}{m+1} - \frac{(3m+8)\cdot\big|B_{m+3}\big|}{24} \right\}, \\
m = 1, 3, 5, \ldots \\
(-1)^{\frac{1}{2}m} \frac{\big|B_{m+2}\big|}{2} < \gamma_m < (-1)^{\frac{1}{2}m} \left\{ \frac{\big|B_{m+2}\big|}{2} - \frac{(m+3)(m+4)\cdot\big|B_{m+4}\big|}{48} \right\}, \\
m = 2, 4, 6, \ldots
\end{gathered} \tag{49}$$

This estimation is relatively tight for moderate values of m, and becomes less and less accurate as m increases. However, even for large m, it remains more accurate than the well-known Berndt's estimation

$$\big|\gamma_m\big| \leqslant \begin{cases} \dfrac{2\,(m-1)!}{\pi^m}, & m = 1, 3, 5, \ldots \\[2ex] \dfrac{4\,(m-1)!}{\pi^m}, & m = 2, 4, 6, \ldots \end{cases} \tag{50}$$

see [16, pp. 152–153], more accurate than Lavrik's estimation $|\gamma_m| \leqslant m!\,2^{-m-1}$, see [113, Lemma 4], more accurate than Israilov's estimation

$$|\gamma_m| \leqslant \frac{m!\,C(k)}{(2k)^m} \tag{51}$$

where k is an arbitrary chosen positive integer and $C(1) = \frac{1}{2}$, $C(2) = \frac{7}{12}$, $C(3) = \frac{11}{3}, \ldots,$ see [84,7], and more accurate than Nan-You–Williams' estimation

$$\bigl|\gamma_m\bigr| \leqslant \begin{cases} \dfrac{2\,(2m)!}{m^{m+1}(2\pi)^m}\,, & m = 1, 3, 5, \ldots \\ \dfrac{4\,(2m)!}{m^{m+1}(2\pi)^m}\,, & m = 2, 4, 6, \ldots \end{cases} \tag{52}$$

see [128, pp. 148–149]. Besides, our estimation also contains a sign, while above estimations are signless. At the same time, (49) is worse than Matsuoka's estimation [121,122], $|\gamma_m| < 10^{-4} \ln^m m$, $m \geqslant 5$, which, as far as we know, is currently the best known estimation in terms of elementary functions for the Stieltjes constants.[39] Note, however, that estimation's bounds (49) may be bettered if we transform parent series (46) into a less divergent series, provided the new series remains enveloping.

Appendix A. Two simple integral formulæ for the Stirling numbers of the first kind

A.1. The first formula

Consider equation (7a) defining signless Stirling numbers of the first kind. Dividing both sides by z^{k+1}, where $k = 1, 2, 3, \ldots,$ and integrating along a simple closed curve L encircling the origin in the counterclockwise direction, we have

$$\oint\limits_L \frac{(z)_n}{z^{k+1}}\,dz = \sum_{l=1}^{n} \bigl|S_1(n,l)\bigr| \cdot \underbrace{\oint\limits_L z^{l-k-1}\,dz}_{2\pi i\,\delta_{l,k}} = 2\pi i\,\bigl|S_1(n,k)\bigr| \tag{A.1}$$

in virtue of Cauchy's theorem. In practice, it is common to take as L the unit circle, and hence

[39] Numerical simulations suggest that Matsuoka's estimation [121,122] may be considerably improved, see e.g. [108]. Recently, Knessl and Coffey reported that they succeeded to significantly better Matsuoka's estimation and even to predict the sign of γ_m. The authors published their findings in [98], and also reprinted them in [97, Theorem 1]. We, however, were not able to verify these results, because several important details related to $v(n)$ from pp. 179–180 [98] were omitted. Estimation of the similar nature was later proposed by Adell [7], but Saad Eddin [150, Tab. 2], [151] reported that Adell's estimation may provide less accurate results than Matsuoka's estimation. Saad Eddin also provides an interesting estimation for the Stieltjes constants, see [150, Tab. 2] and mentions some further works related to the estimations of the derivatives of certain L-functions. Yet, very recently we found another work devoted to the estimation of Stieltjes constants [55]; the latter resorts to the Lambert W-function.

$$\left|S_1(n,k)\right| = \frac{1}{2\pi i}\oint\limits_{|z|=1}\frac{(z)_n}{z^{k+1}}\,dz = \frac{1}{2\pi}\int\limits_{\alpha-\pi}^{\alpha+\pi}\frac{\Gamma(n+e^{i\varphi})}{\Gamma(e^{i\varphi})}\,e^{-i\varphi k}\,d\varphi \tag{A.2}$$

where $\alpha \in \mathbb{R}$.

Alternatively, in virtue of the Cauchy residue theorem, we also have

$$\left|S_1(n,k)\right| = \operatorname*{res}_{z=0}\frac{\Gamma(n+z)}{z^{k+1}\Gamma(z)}\,, \qquad n,k = 1,2,3,\ldots \tag{A.3}$$

A.2. The second formula

Consider generating equation (8a) for the unsigned Stirling numbers of the first kind. Proceeding as above and then making a change of variable $z = re^{i\varphi}$, we obtain

$$\left|S_1(n,k)\right| = \frac{(-1)^k}{2\pi i}\cdot\frac{n!}{k!}\oint\limits_{|z|=r}\frac{\ln^k(1-z)}{z^{n+1}}\,dz = \frac{(-1)^k}{2\pi}\cdot\frac{n!}{k!}\int\limits_{\alpha-\pi}^{\alpha+\pi}\frac{\ln^k\left(1-re^{i\varphi}\right)}{r^n}\,e^{-i\varphi n}\,d\varphi \tag{A.4}$$

where, due to the radius of convergence of (8a), $0 < r < 1$, and $\alpha \in \mathbb{R}$. Obviously, the line integral in the middle may be taken not only along the indicated circle, but along any simple closed curve encircling the origin in the counterclockwise direction and lying inside the unit circle $|z| = 1$. By the way, it is interesting that the same integral taken between $(0,1)$ reduces to a finite linear combination of Stirling numbers of the first kind and ζ-functions, see [18, Sect. 2]. It seems also appropriate to note here that several slightly different integral formulæ of the same kind as (A.4) were given by Dingle [50, pp. 92, 199]. The author, however, did not specify the integration contour. This inaccuracy has been partially corrected by Temme [174, p. 237], who indicated that the integration contour should be a “small circle”.[40]

Similarly to the previous case, the signless Stirling numbers of the first kind may be also given by the following residue

$$\left|S_1(n,k)\right| = (-1)^k\frac{n!}{k!}\operatorname*{res}_{z=0}\frac{\ln^k(1-z)}{z^{n+1}}\,, \qquad n,k = 1,2,3,\ldots \tag{A.5}$$

Finally, expression (A.4) may be also used to readily get an interesting bound for Stirling numbers. In fact, one may notice that if $r \leqslant 1-e^{-1} \approx 0.63$, then the principal branch $\left|\ln^k(1-z)\right| \leqslant 1$ independently of k. Therefore, for $k = 1,2,\ldots,n$

[40] We, however, note this condition is not really necessary; it is sufficient that the integration path be a simple closed curve encircling the origin in the right direction and lying inside the unit disc. This remark may be important for the numerical evaluation of these integrals, which have been reported as not well-suited for these purposes.

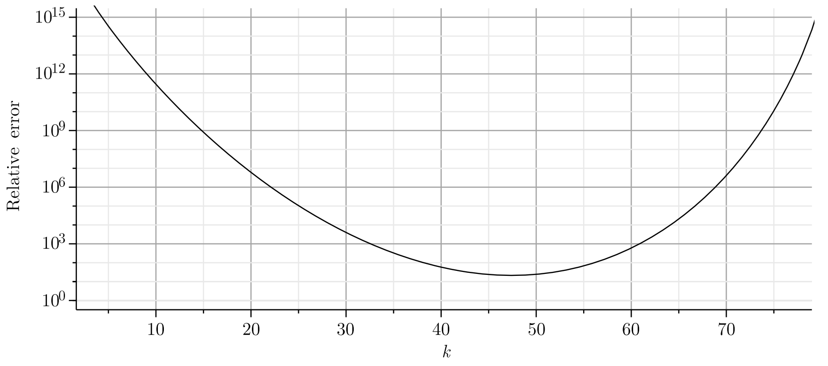


Fig. A.5. Relative error of the estimation for the Stirling numbers of the first kind given by (A.6) as a function of k for $n = 80$, logarithmic scale.

$$\bigl|S_1(n,k)\bigr| \leqslant \frac{n!}{2\pi\, k!}\oint\limits_{|z|=r}\left|\frac{\ln^k(1-z)}{z^{n+1}}\right|\cdot\bigl|dz\bigr| \leqslant \frac{n!}{\bigl(1-e^{-1}\bigr)^n\, k!} \tag{A.6}$$

since $\bigl|dz\bigr| = r\,d\varphi$. If n is prescribed, this estimation is relatively rough for small and large (close to n) factors k; in contrast, for values of k which are slightly greater than $n/2$, this estimation is quite accurate, see Fig. A.5.

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Corrigendum

Corrigendum to "Expansions of generalized Euler's constants into the series of polynomials in π^{-2} and into the formal enveloping series with rational coefficients only" [J. Number Theory 158 (2016) 365–396]

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The author regrets the following inaccuracies in the published article.

- P. 370, footnote 13, second line: "contained an error and" should be omitted. A careful reading of [56] reveals that Franel did not consider the case $m = 0$ and used formula (11) only for $\gamma_1, \gamma_2, \gamma_3, \ldots$. So, it is true that original Franel's formula is not valid for $m = 0$, but he never treated such a case.
- P. 387, Sect. 3.4, first line: "(47)" should read "(46)".

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- P. 387, Sect. 3.4, formulæ (49). These inequalities provide the interval in which the mth Stieltjes constant γ_m lies. The interval itself is correct, but for $m = 3, 4, 7, 8, 11, 12, \ldots$, the left and right parts should be interchanged. Thus, accounting for the sign of $(-1)^{\frac{1}{2}(m+1)}$ and $(-1)^{\frac{1}{2}m}$, these inequalities may be explicitly written as

$$
\begin{aligned}
&-\frac{|B_{m+1}|}{m+1} < \gamma_m < \frac{(3m+8)\cdot|B_{m+3}|}{24} - \frac{|B_{m+1}|}{m+1}, && m = 1, 5, 9, \ldots \\
&\frac{|B_{m+1}|}{m+1} - \frac{(3m+8)\cdot|B_{m+3}|}{24} < \gamma_m < \frac{|B_{m+1}|}{m+1}, && m = 3, 7, 11, \ldots \\
&-\frac{|B_{m+2}|}{2} < \gamma_m < \frac{(m+3)(m+4)\cdot|B_{m+4}|}{48} - \frac{|B_{m+2}|}{2}, && m = 2, 6, 10, \ldots \\
&\frac{|B_{m+2}|}{2} - \frac{(m+3)(m+4)\cdot|B_{m+4}|}{48} < \gamma_m < \frac{|B_{m+2}|}{2}, && m = 4, 8, 12, \ldots
\end{aligned}
\tag{49}
$$

  Case $m = 4, 8, 12, \ldots$ may be also extended to $m = 0$, if we recall that in this case it gives bounds for $\gamma - \frac{1}{2}$, see (46). This case yields $\frac{23}{40} < \gamma < \frac{7}{12}$, which is undoubtedly true. Note also that bounds (49) are always rational, which may be of interest in certain circumstances.
- P. 388, after Eq. (51): "where k is an arbitrary chosen positive integer and $C(1) = \frac{1}{2}$, $C(2) = \frac{7}{12}$, $C(3) = \frac{11}{3}, \ldots$" should read "for $k = 1, 2, 3$, where $C(1) = \frac{1}{2}$, $C(2) = \frac{7}{12}$, $C(3) = \frac{11}{3}$,".
- P. 389, in the middle of the page: "[18, Sect. 2]" should read "[18, Sect. 2.2, Eq. (48)]".
- P. 391, in Ref. [18]: "Math. Comp. (2015), in press" should read "J. Math. Anal. Appl. 442 (2016) 404–434".