Analytic Method for the Computation of the Total Harmonic Distortion by the Cauchy Method of Residues

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***Abstract*—The total harmonic distortion (THD) is an important performance criterion for almost any communication device. In most cases, the THD of a periodic signal, which has been processed in some way, is either measured directly or roughly estimated numerically, while analytic methods are employed only in a limited number of simple cases. However, the knowledge of the theoretical THD may be quite important for the conception and design of the communication equipment (e.g. transmitters, power amplifiers). The aim of this paper is to present a general theoretic approach, which permits to obtain an analytic closed–form expression for the THD. It is also shown that in some cases, an approximate analytic method, having good precision and being less sophisticated, may be developed. Finally, the mathematical technique, on which the proposed method is based, is described in the appendix.**

***Index Terms*—Total harmonic distortion (THD), harmonic analysis, signal analysis, Fourier series, analytic methods, theoretic techniques, complex analysis, Cauchy's theorem, residue theorem, continuous-time filters, transmitters, power amplifiers.**

I. Introduction

The total harmonic distortion (THD) is an important performance criterion for versatile communication devices. The lesser the THD, the closer the signal's spectrum to the ideal one and the lesser the interferences for other electronic equipment. Moreover, the problem of distorted and not eco–friendly radio–emissions is becoming more and more important, especially, in the quite recent paradigm of spectrum sharing and spectrum sensing.

In practice, transmitted signals are often obtained from periodic ones of standard form via filtering. For instance, class C and D power amplifiers are nowadays very popular in modern communication devices (especially, in their transmitting circuits). These amplifiers are more efficient than traditional class A, AB and B amplifiers[1], but the other side of the coin is their high non–linearity. Their output signal is very "dirty" and "noisy" (in the sense that it contains too many harmonics of the fundamental frequency), and even if the input signal was sinusoidal, the output one is much closer to the pulse train, which needs to be filtered before it goes to the antenna. To this end, LRC filtering networks, ceramic or cavity filters are usually inserted between the output transistor/tube and antenna. Another example may be the use of the motherboard clock frequency as the carrier for a transmitter. In this case, the square waveform of fundmental period T should be converted into a sinusoidal one of the same period T (see e.g. [7]). Also, some FM stereo transmitters use the square signal instead of the sinusoidal one as stereo sub–carrier (at 38 kHz). Another exampe: it may be sometimes necessary to obtain a nth sub–harmonic of a sinusoidal signal ($n \in \mathbb{N}^*$). For example, in FM stereo modulators, a first sub–harmonics of a stereo sub–carrier (which is at 38 kHz) is added to the composite stereo signal; it is called *a 19kHz pilot–tone*[2] [8]. Technically, the latter task is quite sophisticated for the sinusoidal signal, but elementary for the square one. For this reason, the following procedure is often performed: the sinusoidal signal is first transformed into a square waveform, then, its fundamental frequency is devided by n, by means of digital frequency dividers [5], and then, the square signal is filtered in order to turn back to the sinusoidal waveform.

In all aforementioned situations, the T–periodic non–sinusoidal signal is filtered by a band–pass or low–pass filter in order to get the output signal as close as possible to the sinusoidal one of the same period T, see Fig. 1. Mathematically, the latter criterion (to be as close as possible to the sine wave) is usually expressed in terms of the THD, which shows the degree in which the sinewave signal of the fundamental frequency is "noised" by all its harmonics together.[3] The value of the THD is usually obtained either empirically by corresponding measures (e.g., with the help of total harmonic distortion meters and analyzers), [6], [7], [9], or it may be given approximately by making use of various numerical methods, or by using hybrid numerico–empirical THD analyzers (e.g. FPGA–based [10]). Unfortunately, analytic methods are rarely used for this aim,[4] and the main

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[1]The effeciency of class C and class D amplifiers may be up to 80% and 95% respectively, while it does not exceed 35% for class A and 50% for class B amplifiers [1]–[6]. For this reason, nowadays, many portable modern communication devices (e.g. Wi–Fi, GSM, 3G devices, some trancievers, *etc.*) are equipped with class C and D amplifiers.

[2]Moreover, the RDS component sub–carrier, placed at 57 kHz, is the third harmonics of the pilot–tone, and there were also in US some SCA broadcasting services using higher harmonics of the pilot–tone.

[3]Note, by the way, that the THD can be also interpreted as the inverse of the signal–to–noise ratio (SNR), if harmonics are regared as noise and the fundamental sinewave as signal of which the SNR is calculated.

[4]One may mention works [11]–[22], where various efforts for the analytic evaluation of the THD were enterprised (see also [23], [24] in which a connection between intermodulation noise and THD is addressed). However, none of them is fully analytic in the sense that the infinite THD sums (9) are always computed numerically. The unique exception is the publication [22], were a formula from [25] was used to evaluate analytically the THD sum. One may also mention work [11], where the convergence of the THD series was studied analytically.

reason for this, as was noticed in [22], is that the accurate analytic evaluation of the THD is, in general, quite difficult. As a result, many authors try to circumvent this difficulty by using numerical methods. Basically, these methods truncate the infinite THD sum [see formula (9)] at some term, after which the total contribution of all remaining terms is believed to be small enough; then, the gathered terms are banally summed. For example, in [26] the THD sum is truncated at 49th harmonics, in [10] at 50th, in [16] at 63rd harmonics, in [17] at 200th harmonics. Notwithstanding, the knowledge of the theoretical THD may be important not only from analytic and methodogical points of view, but also it may be crucial for the conception and design of versatile electronic devices. The aim of this paper is to provide a fully analytic method for calculation of the THD of non–sinusoidal signals with known Fourier coefficients filtered by a band–pass or low–pass filter with given parameters (e.g. the Q–factor, the filter's order, *etc.*). Of course, this includes also cases, where the signal was not filtered, and simply, the analytic calculation of the THD of a given signal is of interest. Moreover, we try to provide, wherever possible, two analytic methods for this aim: an approximate one, which is usually not sophisticated and provides good accuracy, and the exact one, which is more sophisticated and is based on the theory of functions of a complex variable, in particular, on a Cauchy method of residues.

II. Generalities on the Evaluation of the THD

A. Preliminaries

Without loss of generality, we assume that the continuous T–periodic non–sinusoidal real–valued signal $x(t)$ can be expanded in Fourier series. We recall that, such expansion, in a very general way, can be written in the following form (see e.g., [27]–[30]):

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos\omega_0 nt + b_n\sin\omega_0 nt\big)\,, \qquad (1)$$

where

$$a_k = \frac{2}{T}\int\limits_{t_0}^{t_0+T} x(\tau)\cos\omega_0 k\tau\, d\tau, \quad b_k = \frac{2}{T}\int\limits_{t_0}^{t_0+T} x(\tau)\sin\omega_0 k\tau\, d\tau, \qquad (2)$$

$\omega_0 = 2\pi/T$ is the fundamental circular frequency (called also pulsatance), $k\in\mathbb{N}$, $t_0\in\mathbb{R}$, and $a_0/2$ is the DC component present in the signal $x(t)$ [more precisely, it is a time average of $x(t)$ over fundamental period]. This expansion is called trigonometrical Fourier series. It can be easily rewritten in a more compact way, called exponential Fourier series:

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{i\omega_0 nt}\,, \qquad c_k = \frac{1}{T}\int\limits_{t_0}^{t_0+T} x(\tau)\, e^{-i\omega_0 k\tau} d\tau\,, \quad (3)$$

$k\in\mathbb{Z}$. Since both forms are in frequent use, all derivations and most important results will be given for both of them. Note that as we considered that $x(t)\in\mathbb{R}$, Fourier coefficients $a_k\in\mathbb{R}$, $b_k\in\mathbb{R}$, $c_k\in\mathbb{C}$ and $c_k = c^*_{-k}$, where * denotes complex conjugate. It may be also useful to note that $|c_k| = |c_{-k}| = \frac{1}{2}\sqrt{a_k^2+b_k^2}$, $k\in\mathbb{N}$. There is also an important equality for the Fourier series, called Parseval's identity:

$$P_{\text{tot}} = \frac{1}{T}\int\limits_{t_0}^{t_0+T} x^2(\tau)\, d\tau = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\big(a_n^2+b_n^2\big) = \sum_{n=-\infty}^{+\infty}|c_n|^2, \qquad (4)$$

which is a kind of conservation law for the power in time and frequency domains [P_{tot} being a mean power over period of the signal $x(t)$]. From functional point of view, this identity is simply the Pythagorean theorem in the space created by the orthonormal set of exponential functions, into which the signal $x(t)$ is decomposed. Existence of the equality (4) implies that $x(t)\in L^2(T)$ and a_k, b_k and c_k are in $\mathbb{L}^2$. Also, since all Fourier coefficients depend on function $x(t)$ as functionals, we will sometimes write $a_k[x(t)]$, or simply $a_k[x]$, in order to avoid any confusion about the signal whose coefficients are presented (notation $[\,\cdot\,]$ will be always used for functional dependencies).

Finally, throughout the paper, following abbreviated notations are used: $\operatorname{tg} z$ for tangent of z, $\operatorname{ctg} z$ for cotangent of z, $\operatorname{ch} z$ for hyperbolic cosine of z, $\operatorname{sh} z$ for hyperbolic sine of z, $\operatorname{th} z$ for hyperbolic tangent of z, $\operatorname{cth} z$ for hyperbolic cotangent of z, $\zeta(z)$ for Riemann zeta–function of argument z (see, e.g., [31]–[34]). $\mathrm{Re}[z]$ and $\mathrm{Im}[z]$ denote respectively real and imaginary parts of z; letter i is never used as index and is $\sqrt{-1}$; $\operatorname*{res}_{z=a} f(z)$ stands for the residue of the function $f(z)$ at the point $z=a$; other notations are standard.

B. THD of Periodic Signals of Standard Waveforms

Consider, for instance, the T–periodic square wave of amplitude A, given by the equation $x_{\text{sq}}(t) = A\,\mathrm{sgn}(\sin\omega_0 t)$ (see also Fig. 1). It may be expanded in Fourier series as follows:

$$x_{\text{sq}}(t) \;=\; \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin\omega_0 t(2n-1)}{2n-1}\,. \qquad (5)$$

The T–periodic triangle wave of amplitude A, given formally by the equation $x_{\text{tr}}(t) = (2A/\pi)\arcsin(\sin\omega_0 t)$, may be written as:

$$x_{\text{tr}}(t) \;=\; \frac{8A}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\omega_0 t(2n-1)}{(2n-1)^2}\,. \qquad (6)$$

The sawtooth signal of period T of amplitude A, which may be regarded as $x_{\text{sw}}(t) = (2A/\pi)\operatorname{arctg}(\operatorname{tg}\omega_0 t)$, has the following representation in Fourier series:

$$x_{\text{sw}}(t) \;=\; \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin\omega_0 tn}{n}\,. \qquad (7)$$

The rectangular pulse wave with cyclic ratio $\mu\in(0,1)$, i.e. the pulse train (see Fig. 2), which is the typical signal at the output of class D amplifiers, admits the following Fourier expansion:

$$x_{\text{pt}}(t,\mu) = A(2\mu-1) + \frac{2A}{\pi}\sum_{n=1}^{\infty}\bigg(\frac{\sin 2\pi\mu n}{n}\cos\omega_0 nt + \frac{1-\cos 2\pi\mu n}{n}\sin\omega_0 nt\bigg)\,; \qquad (8)$$

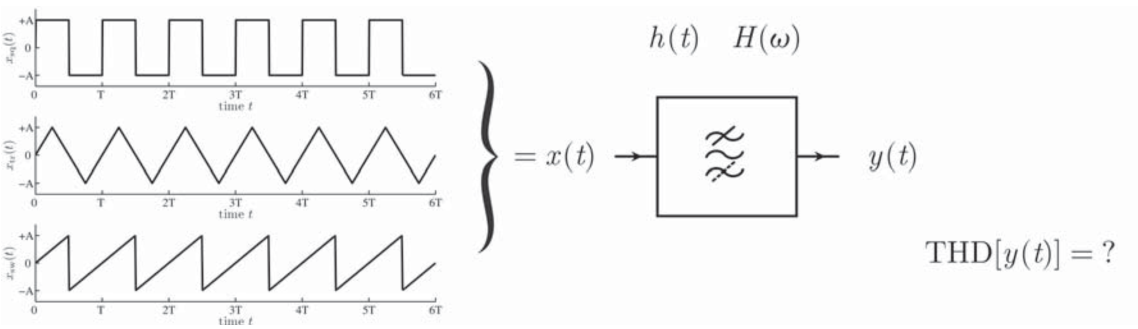

Fig. 1. Filtering of some frequent periodic signals of standard form of period T and of amplitude A. Analytic calculation of the THD of the output signal $y(t)$ is the interest of the study.

and so on... (see e.g. [27], [28], [35], or some Fourier series handbook for a more complete list of Fourier expansions).

From the Fourier expansions, one may compute corresponding THDs with respect to the sinusoidal wave of the fundamental angular frequency ω_0. The amplitude of the kth harmonics of the frequency ω_0 in the signal $x(t)$ is represented by $2|c_k| = \sqrt{a_k^2 + b_k^2}$. Since the THD is the square root of the ratio of the sum of the powers of all harmonic components, denoted by P_{har} to that of the fundamental frequency, designated by P_{sig}, the THD may be expressed in terms of Fourier coefficients:

$$\text{THD}[x(t)] = \sqrt{\frac{P_{\text{har}}}{P_{\text{sig}}}}$$

$$= \sqrt{\frac{\sum_{n=2}^{\infty}\left(a_n^2 + b_n^2\right)}{a_1^2 + b_1^2}} = \sqrt{\frac{\sum_{n=2}^{\infty}|c_n|^2}{|c_1|^2}}\,. \tag{9}$$

For the square waveforme given by (5), we have

$$a_n = 0\,, \quad b_{2n-1} = \frac{4A}{\pi(2n-1)}\,, \quad b_{2n} = 0\,, \tag{10}$$

where $n \in \mathbb{N}$; formula (9) therefore yields:

$$\text{THD}[x_{\text{sq}}(t)] = \sqrt{\frac{\pi^2}{8} - 1} \approx 48.3\%. \tag{11}$$

To obtain this results we used the well–known summation formula[5]

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}\,, \tag{12}$$

which is a particular case of a more general formula:

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = \pi^{2k}\frac{(2^{2k}-1)\cdot|B_{2k}|}{2\cdot(2k)!}\,, \qquad k \in \mathbb{N}^*\,, \tag{13}$$

where B_n are so–called Bernoulli numbers. The first few are: $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_{2n+1} = 0$ for $n \in \mathbb{N}^*$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$, $B_{12} = -691/2730$, *etc.* Similar formulæ exist also for series involving general terms , $1/n^{2k}$, $(-1)^k/n^{2k}$, $1/(2n-1)^{2k-1}$, $1/n^p$ with $p \notin \mathbb{N}$, *etc.*; they contain Bernoulli numbers, Euler numbers, Catalan's constant, Riemann ζ–function, and other special numbers and functions (details may be found in [27], [28], [34], [36]–[38]). In the analogous manner, we may compute the THD of some other signals. For instance, for the triangle wave, we have:

$$\text{THD}[x_{\text{tr}}(t)] = \sqrt{\frac{\pi^4}{96} - 1} \approx 12.1\%, \tag{14}$$

with the help of (13) for $k = 2$. The same procedure for the sawtooth signal (7) yields:

$$\text{THD}[x_{\text{sw}}(t)] = \sqrt{\frac{\pi^2}{6} - 1} \approx 80.3\%, \tag{15}$$

by taking into account that $\zeta(2) = \pi^2/6$.

For the rectangular pulse train (8), the things are slightly different. For this signal we cannot use tabular formula (13) and similar ones; we should rather use its Parseval's identity

$$\frac{\mu(1-\mu)\,\pi^2}{2} = \sum_{n=1}^{\infty} \frac{\sin^2 \pi\mu n}{n^2}\,, \qquad \mu \in (0,1)\,, \tag{16}$$

which is also a generalization of (12).[6] Thus, we obtain for the THD of $x_{\text{pt}}(t,\mu)$ the following result:

$$\text{THD}[x_{\text{pt}}(t,\mu)] = \sqrt{\frac{\mu(1-\mu)\pi^2}{2\sin^2 \pi\mu} - 1}\,. \tag{17}$$

It is depicted in Fig. 2, and it logically reaches the minimum when the signal $x_{\text{pt}}(t,\mu)$ becomes "symmetric", i.e. when $\mu = 0.5$. The THD of such signal is symmetric about $\mu = 0.5$, i.e. $\text{THD}[x_{\text{pt}}(t,\mu)] = \text{THD}[x_{\text{pt}}(t,1-\mu)]$.

C. THD of Filtered Periodic Signals of Standard Waveforms

Let the filter's impulse response $h(t)$ be real–valued and Lebesgue integrable function (i.e., the filter is stable). Consider now signal $y(t)$, which is the filtered version of $x(t)$; in other words $y(t) = h(t) * x(t)$. Since $x(t)$ is an $L^2(T)$ signal, so is $y(t)$. The Fourier coefficients of $y(t)$ strongly depend on filter's parameters: each of them is the corresponding coefficient of input signal $x(t)$, weighted by the filter's transfer function at specified frequency. Thus, the amplitude of the kth harmonics at the output of the filter is $2\,\big|c_k[y(t)]\big| =$

[5]It can be also obtained from Parseval's identity (4) for $x_{\text{sq}}(t)$.

[6]Formula (16) reduces to (12) when $\mu = 0.5$.

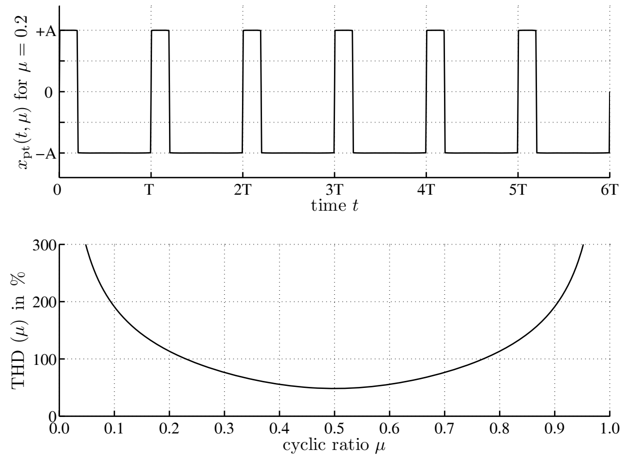

Fig. 2. Top: a T–periodic rectangular pulse train of amplitude A and with cyclic ratio (duty cycle) μ. Bottom: its THD as a function of cyclic ratio.

$2\left|H(k\omega_0)\right|\cdot\left|c_k[x(t)]\right|,\ k\in\mathbb{Z}$. The THD of the output signal therefore reads:

$$\mathrm{THD}[y(t)] = \sqrt{\frac{\displaystyle\sum_{n=2}^{\infty}\big(a_n^2[x]+b_n^2[x]\big)\left|H(n\omega_0)\right|^2}{\big(a_1^2[x]+b_1^2[x]\big)\left|H(\omega_0)\right|^2}}$$

$$= \sqrt{\frac{\displaystyle\sum_{n=2}^{\infty}\left|c_n[x]\,H(n\omega_0)\right|^2}{\left|c_1[x]\,H(\omega_0)\right|^2}}\,. \qquad (18)$$

Since $y(t)\in L^2(T)$, by Parseval's identity Fourier coefficients $c_k[y(t)]$ are in $\mathbb{L}^2$; consequently, latter sums always converge. It is quite clear that such sums may be difficult to evaluate analytically. Of course, there are no doubts that such sums may be computed by various numerical methods, but we would like to get an exact theoretical result. For the signal $x(t)$ we could calculate infinite sums in THD expressions thanks to some tabular formulæ, or to the Parseval's identity. But, it is obvious, that on the one hand, there can not be such formulæ for all occasions, and not always a sum can be reduced to one from handbook. Whereupon, there is only Parseval's identity for the signal $y(t)$ that can provide such sums. On the other hand, for the use of the Parseval's identity for this aim, $y(t)$ should be known, and the latter implies the knowledge of the impulse response of the filter $h(t)$ or of its transfer function $H(\omega)$. More precisely, $H(\omega)$ should be completely known (i.e. we know its absolute value and phase). However, in many practical cases, we only know its absolute value $|H(\omega)|$ (for example, obtained by direct measurement as it is often the case for quartz–crystal, electromechanical, complex LRC networks, ceramic filters, *etc.*, see e.g. [1]–[3], [39]–[45]). Moreover, as we come to see later, we do not even need $|H(\omega)|$, but only its values at frequencies $k\omega_0$, $k\in\mathbb{N}$, which is even a more weak condition.

More generally, the problem lies in fact that sums of series, unlike integrals, are not so tabularized in literature (see e.g. [27], [28], [35]–[37], [46]–[48]). Furthermore, there are not so many classic methods for the summation of series; see e.g. [49, pp. 225–287], [50, pp. 293–321], [51, pp. 7–36], [52, pp. 255–272], [53]. If, for instance, we slightly modify formula (12) by replacing its general term by $1/(\alpha n^2+\beta n+\gamma)$, where α, β and γ are arbitrary real non–integers constants, the corresponding series will be convergent and numerical methods can provide the convergence point, but it is not listed in mathematical handbooks, and it can not be reduced to some tabular series. So, is it possible to arrive to an analytic closed–form expression for such kind of non–tabular series, and if yes, how one can get this result? Despite the recent decline of analytic methods in favor of numeric ones, the answer is, in many cases, 'yes, it is'. In particular, we will present, in this paper, an interesting method, based on the Cauchy theorem from complex variable theory, that can be precisely applied for the analytic calculation of the non–tabular series, and which permits, *inter alia*, to obtain the closed–form expression for the THD. The latter theorem, proved by Augustin–Louis Cauchy in the first half of XIXth century, is mainly used for the evaluation of contour or improper integrals. In signal processing and related disciplines, it is especially employed for the calculation of the inverse Fourier, Laplace, Hilbert, Mellin and Z transforms, whereas in other domains this theorem has quite limited use. Among such uses, one may mention works in following fileds: error–rate evaluation in fading communication channels [54]–[56], non–uniform sampling [57], calculation of eigenmodes without SVD technique [58], applications for Bode's theorem [59], analysis of data transmission over digital networks [60], [61], and some others [62]–[67].

Since the aforementioned method is not directly related to the subject of this paper, we put it in the appendix. Note also that the method is quite general; under some mild restrictions on the summed function, which are generally satisfied whenever the series converges, it permits to deal analytically with many finite and infinite series: from simple energy–like sums to discrete Fourier transforms, trace and determinant calculations, kernel expansions in eigenvalues and eigenfunctions, and many others. Finally, the readers, who are interested in other special methods and further reading devoted to the summation of series, might also appreciate following references: [68]–[74].

III. Application of the Residue Method to the Analytic Evaluation of the THD

A. Introductory Remarks

In order to use formula (44), we must, first, be sure that the conditions on summed function $\left|H(k\omega_0)\,c_k[x]\right|^2$ are satisfied, and then, reduce the sum in the numerator of (18) to that over the integers from $-\infty$ to $+\infty$. The former is usually true in cases like ours [all considered signals are in $L^2(T)$], because harmonics' amplitude $2|c_k[x]|$ decrease with k, and analog filters are band– or low–pass, absolute value of transfer function of which decrease with frequency as well. In addition, function $\left|c_z[x]\,H(z\omega_0)\right|$ is normally regular everywhere in a complex plane, except for a finite number of isolated singularities. As to the latter, it is elementary since $|c_{-k}|=|c_k|$.

Formula (44) is the key of the proposed method: infinite sum was reduced the the finite one containing only m terms, where m is usually not grater than ten (it is closely related to the filter's topology and to its order).

B. THD of the Square Wave Filtered by a Band–Pass Filter

Consider a transfer function of a typical band–pass filter, whose absolute value may be written as:

$$\big|H(\omega)\big| = \frac{H_0}{\sqrt{1+Q^2\left(\dfrac{\omega}{\omega_r}-\dfrac{\omega_r}{\omega}\right)^2}}\,, \qquad H_0,\omega_r,Q>0\,, \tag{19}$$

where ω_r is the resonant frequency, H_0 is the maximum gain of the transfer function at the resonant frequency, Q is the quality factor, commonly abbreviated to Q–factor. Obviously, since we wish to select fundamental frequency in the periodic signal $x(t)$, filter's resonant frequency $\omega_r = \omega_0$. Given the signal x_{sq} [see (5)], whose Fourier coefficients are given explicitely in (10), and given the absolute value of transfer function (19), we get the absolute value of the Fourier coefficients for the filtered signal $y(t)$:

$$\big|c_k[y]\big|^2 = \big|c_k[x]\,H(k\omega_0)\big|^2 =$$

$$\begin{cases} 0\,, & k=2n\,;\\[2mm] \dfrac{4A^2H_0^2\pi^{-2}}{Q^2(2n-1)^4-(2Q^2-1)(2n-1)^2+Q^2}\,, & k=2n-1\,; \end{cases}$$

where $n\in\mathbb{N}^*$. The explicit expression for the THD of the filtered signal, given by (18), therefore reads: $\text{THD}[y(t)] =$

$$= \sqrt{\sum_{n=2}^{\infty}\frac{1}{Q^2(2n-1)^4-(2Q^2-1)(2n-1)^2+Q^2}} \tag{20}$$

1) Approximate Calculation of the THD for Medium and Great Q–Factors: Let $Q \gg 0.7$; then $2Q^2-1\approx 2Q^2$. In this case, the last formula may be simplified as follows:

$$\text{THD}[y(t)] \approx \frac{1}{4Q}\sqrt{\sum_{n=2}^{\infty}\frac{1}{n^2(n-1)^2}}\,, \qquad Q\gg\frac{1}{\sqrt{2}}\,.$$

This expression is quite simple, and the desired sum[7] may be reduced to tabular ones by partial fraction decomposition:

$$\sum_{n=2}^{\infty}\frac{1}{n^2(n-1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)^2} \tag{21}$$

$$= -2\underbrace{\sum_{n=1}^{\infty}\frac{1}{n(n+1)}}_{1} + \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\pi^2/6} + \underbrace{\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}}_{\pi^2/6\,-\,1} = \frac{\pi^2}{3}-3\,.$$

The final result is straightforward:

$$\text{THD}[y(t)] \approx \frac{1}{4Q}\sqrt{\frac{\pi^2}{3}-3} \approx \frac{0.1346}{Q}\,, \quad Q\gg\frac{1}{\sqrt{2}}\,. \tag{22}$$

The THD, as a function of Q, is depicted in Fig. 3, together with that computed exactly by residue method (obtained in the next subsection).

[7]See also exercice n° 2997 [49, p. 275 and p. 504].

2) Exact Calculation of the THD for Arbitrary Q–Factors: In previous subsection, we used an approximate method for the evaluation of the formula (20). Of course, it was a chance that the needed series could be reduced to the tabular ones, and in general, it is not possible; besides, the calculation was approximate. We now calculate it exactly by using the aforementioned residue method.

Let us denote, for simplicity, the general term of the sum in (20) by

$$f(k) \equiv \frac{1}{Q^2(2k-1)^4-(2Q^2-1)(2k-1)^2+Q^2}\,.$$

In order to use the residue method for the calculation of the infinite sum in (20), we must verify that the conditions on the summed function are satisfied.

1) As $|z|\to\infty$,

$$|f(z)| \approx \frac{1}{16Q^2|z|^4} < \frac{C}{|z|^\xi}\,, \qquad \xi>1\,. \tag{23}$$

2) Is the function $f(z)$ regular for $z\in\mathbb{Z}$? To respond to this question, we must locate its singularities. Function $f(z)$ has four simple poles, in terms of which it may be written as:

$$f(z) = \frac{1}{16Q^2(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

where the poles are:

$$z_{1,2,3,4} = \frac{1}{2}\pm\frac{1}{2}\sqrt{1-\frac{1}{2Q^2}\pm\sqrt{\frac{1-4Q^2}{4Q^4}}}\,, \tag{24}$$

For convenience, we introduce an auxiliary function μ:

$$\mu \equiv \sqrt{1-\frac{1}{2Q^2}+\sqrt{\frac{1-4Q^2}{4Q^4}}}\,, \tag{25}$$

where we also suppose that $Q>\frac{1}{2}$, which is largely true in practice.[8] Then, the poles of $f(z)$ may be easily written in terms of μ:

$$z_1 = \frac{1+\mu}{2} = z_3^*\,, \qquad z_2 = \frac{1-\mu}{2} = z_4^*\,. \tag{26}$$

Thus, all poles are complex, and therefore, for $z\in\mathbb{Z}$, function $f(z)$ is regular.

3) Since there are only four poles, the number of singularities $m=4$, and except these points, function $f(z)$ is regular everywhere in $\mathbb{C}$.

One may finally notice that $f(z)$ is single–valued. All conditions on $f(z)$ are therefore fulfilled, and consequently, (20) can be computed via the residue method.

By noticing that

$$\sum_{n=2}^{\infty}f(n) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}f(n) - f(1)\,,$$

[8]The case $0<Q\leqslant\frac{1}{2}$ is uninteresting from practical point of view, since filters with such a small quality factor do not attractive. Nevertheless, methematically it can be always solved, but since in this case μ becomes purely imaginary, the location of poles in the complex plane qualitatively change. Consequently, (26) is not true anymore; one should simply use expressions (24) (μ was introduced only for the simplification of final formulæ).

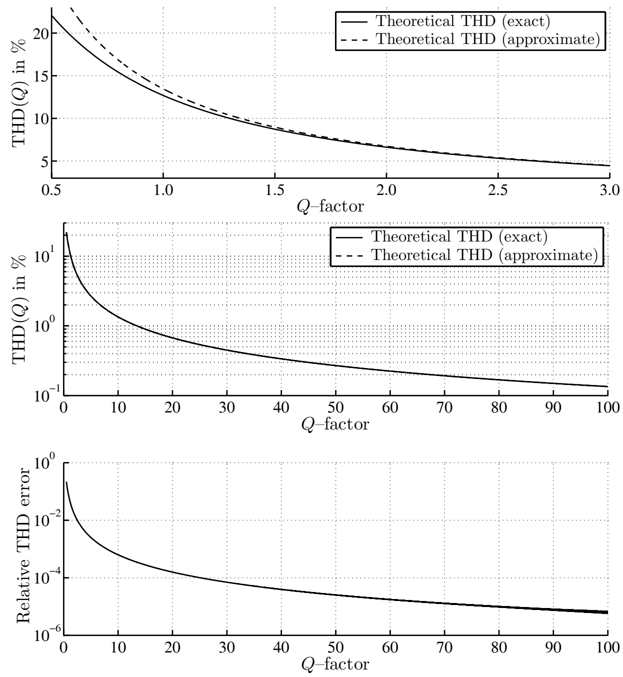

Fig. 3. Top and middle: theoretical THD computed accordingly to the approximate method and to the exact residue–based method. Bottom: relative error between two analytic methods.

and that $f(1) = 1$, we apply equality (44). The THD becomes:

$$\mathrm{THD}[y(t)] = \sqrt{-\frac{\pi}{2}\sum_{l=1}^{4}\operatorname*{res}_{z=z_l}\big[f(z)\operatorname{ctg}(\pi z)\big] - f(1)}$$

$$= \left[\frac{\pi}{8\,Q^2(\mu^{*2}-\mu^2)}\left(\frac{1}{\mu}\operatorname{ctg}\frac{\pi(1+\mu)}{2} - \frac{1}{\mu}\operatorname{ctg}\frac{\pi(1-\mu)}{2}\right.\right.$$

$$\left.\left. + \frac{1}{\mu^*}\operatorname{ctg}\frac{\pi(1-\mu^*)}{2} - \frac{1}{\mu^*}\operatorname{ctg}\frac{\pi(1+\mu^*)}{2}\right) - 1\right]^{\frac{1}{2}} = \ldots$$

$$\ldots = \sqrt{\frac{\pi}{2\sqrt{4Q^2-1}}\,\mathrm{Im}\left[\frac{\sin\pi\mu}{\mu(1+\cos\pi\mu)}\right] - 1}\,, \qquad Q > \frac{1}{2}\,,$$

with μ given by (25). The graph of this function is presented in Fig. 3. From the latter, we may conclude that the THD decrease quite quickly with Q (as $1/Q$ for great Q). Even a filter with Q–factor greater than 14 suffices for the output THD lesser than 1%. If, in contrast, we use some special filters with high Q–factors, such as electromechanical filters (see e.g. [3], [39]–[43], [45]), which are widely used is single–side band (SSB) equipment and high–performance analog transceivers, their Q–factors is usually as high as 10,000–20,000[9] (which are by the way not easy to attempt even with digital filters). In these cases, the THD at their output will be lesser than 0.001%, which is sufficient even for Hi–Fi and Hi–End devices. By the way, one may also note that the precision of the approximate method is very good; both curves are almost undistinguishable. In fact, the error is essential only for small Q, and thus, the approximate method may be practically always used. For $Q > 2.5$ the precision of the approximate method becomes better than 1%; for $Q > 8$, it is better than 0.1%; for $Q > 25$ it is better than 0.01%, *etc.*

C. THD of the Sawtooth Wave Filtered by a Low–Pass Filter

Consider a Butterworth filter of pth order, which is a typical low–pass filter, whose gain is:

$$\big|H(\omega)\big| = \frac{H_0}{\sqrt{1+\dfrac{\omega^{2p}}{\omega_c^{2p}}}}\,, \quad H_0, \omega_c > 0\,, \quad p \in \mathbb{N}^*\,, \quad (27)$$

where H_0 is the maximum filter's gain and ω_c is the cut–off frequency. As before, we put $\omega_c = \omega_0$. For the sawtooth wave x_{sw} given by (7), the absolute value of its Fourier coefficients are: $\big|c_k[x]\big| = A/(\pi k)$, $k \in \mathbb{N}^*$. After filtering, the output Fourier coefficients read:

$$\big|c_k[y]\big|^2 = \big|c_k[x]\,H(k\omega_0)\big|^2 = \frac{A^2H_0^2}{\pi^2k^2(1+k^{2p})}\,, \qquad k \in \mathbb{N}^*\,.$$

By expression (18), the THD becomes:

$$\mathrm{THD}[y(t,p)] = \sqrt{2\sum_{n=2}^{\infty}\frac{1}{n^2(1+n^{2p})}}\,. \quad (28)$$

The infinite sum in the above formula is not tabular. We can try either to reduce it by an approximation to some tabular sum, or to compute it exactly by means of residues.

1) Approximate Calculation of the THD for Great Orders p: Let p be such that $2^{2p} \gg 1$, then:

$$\sum_{n=2}^{\infty}\frac{1}{n^2(1+n^{2p})} \approx \sum_{n=2}^{\infty}\frac{1}{n^{2p+2}} = \zeta(2p+2) - 1$$

$$= \frac{(2\pi)^{2p+2}|B_{2p+2}|}{2\cdot(2p+2)!} - 1\,.$$

The THD is then approximately:

$$\mathrm{THD}[y(t,p)] \approx \sqrt{2\,\zeta(2p+2) - 2}$$

$$= \sqrt{\frac{(2\pi)^{2p+2}|B_{2p+2}|}{(2p+2)!} - 2}\,. \quad (29)$$

The so calculated approximate THD is shown in Fig. 4, together with the exact one computed in the next subsection. Note however, that Bernoulli numbers and Riemann ζ–function are themselves difficult to compute (especially the latter), and in addition, it is much more convenient to have an answer in terms of standard functions (if it exists, and particularly, because it is an exact result). Furthermore, many "calculators" do not have these special numbers and functions built–in.[10]

[9]It may be up to 100,000 for torsional magnesium electromechanical filters and up to 400,000 for quartz–crystal filters in vacuum [44].

[10]For example, in MATLAB v7.2.0.232 (R2006a), the Bernoulli numbers are implemented only in a symbolic way via Maple kernel.

2) Exact Calculation of the THD for Arbitrary Orders p: As in previous example, we first denote for simplicity

$$f(k) \equiv \frac{1}{k^2(1+k^{2p})}\,,$$

which is regular for every $k \in \mathbb{Z}$, except the point $k = 0$, which is a double pole; $f(k)$ should be therefore regularized before it could be extended to $k \in \mathbb{C}$, and before we could apply the residue method.[11] Regularization is performed in the same manner as explained in the Section B1 of the appendix. We replace function $f(z)$ by a $\widetilde{f}(z)$, such that

$$\widetilde{f}(z) = \frac{1}{(z^2+\varepsilon^2)(1+z^{2p})}\,, \quad 0 < \varepsilon \ll 1\,.$$

Now it is regular for each $z \in \mathbb{Z}$, and it may be easily verified that $\widetilde{f}(z)$ satisfies all necessary and sufficient conditions on the summed function for the use of the formula (44). Notice that as $\varepsilon \to 0$, $\widetilde{f}(z) \to f(z)$. The so regularized function $\widetilde{f}(z)$ has $2p+2$ simple poles at $z = z_l$, in terms of which it may be written as:

$$\widetilde{f}(z) = \prod_{l=1}^{2p+2} \frac{1}{z - z_l}\,,$$

where

$$\begin{cases} z_{l+1} = e^{\frac{i\pi(2l+1)}{2p}}\,, & l = 0, 1, \ldots, 2p-1\,. \\ z_{2p+1} = +i\varepsilon\,, \\ z_{2p+2} = -i\varepsilon\,. \end{cases} \tag{30}$$

Function $\widetilde{f}(z)$ is even, and hence

$$\sum_{n=2}^{\infty} \widetilde{f}(n) \;=\; \frac{1}{2}\sum_{n=-\infty}^{+\infty} \widetilde{f}(n) - \widetilde{f}(1) - \frac{1}{2}\widetilde{f}(0)\,.$$

From above equations and by (44), the infinite series in (28) may be reduced to the following expression:

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{1}{n^2(1+n^{2p})} \\ &= \lim_{\varepsilon\to 0} \left\{ -\frac{\pi}{2}\sum_{l=1}^{2p+2} \operatorname*{res}_{z=z_l} \big[\widetilde{f}(z)\operatorname{ctg}(\pi z)\big] - \widetilde{f}(1) - \frac{1}{2}\widetilde{f}(0) \right\} \\ &= -\frac{\pi}{2}\sum_{l=1}^{2p} \operatorname*{res}_{z=z_l} \big[f(z)\operatorname{ctg}(\pi z)\big] + \frac{1}{2}\lim_{\varepsilon\to 0} g\,(\varepsilon, p)\,, \end{aligned} \tag{31}$$

where we designated the part depending on ε by

$$g\,(\varepsilon, p) \;\equiv\; \frac{\pi \operatorname{cth} \pi\varepsilon}{\varepsilon\big(1+(-1)^p\varepsilon^{2p}\big)} - \frac{1}{1+\varepsilon^2} - \frac{1}{\varepsilon^2}\,.$$

The evaluation of the limit for $\varepsilon \to 0$ yields:

$$\lim_{\varepsilon\to 0} g\,(\varepsilon, p) \;=\; \begin{cases} \dfrac{\pi^2}{3}\,, & p = 1; \\ \dfrac{\pi^2}{3} - 1\,, & p = 2, 3, 4, \ldots \end{cases}$$

[11]Note that, each time we deal with the signal $x(t)$, whose Fourier coefficients have term k in the denominator, the regularization procedure has great chances to be demanded.

After the calculation of residues, formula (31) gets its final form:

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n^2(1+n^{2p})} \;&=\; -\frac{\pi}{2}\sum_{s=1}^{2p}\left[\frac{\operatorname{ctg}\pi z_s}{z_s^2} \prod_{\substack{l=1 \\ l\neq s}}^{2p} \frac{1}{z_s - z_l} \right] \\ &+ \frac{\pi^2}{6} - \frac{1}{4}\operatorname{sgn}\!\left(p - \frac{3}{2}\right) - \frac{1}{4}\,, \end{aligned}$$

where points $\{z_s\}_{s=1}^{2p}$ are given in the first line of (30). From this formula, analytic expression for the THD is straightforward: $\mathrm{THD}[y(t,p)] \;=$

$$\sqrt{-\pi\sum_{s=1}^{2p}\left[\frac{\operatorname{ctg}\pi z_s}{z_s^2} \prod_{\substack{l=1 \\ l\neq s}}^{2p} \frac{1}{z_s - z_l} \right] + \frac{\pi^2}{3} - \frac{1}{2}\operatorname{sgn}\!\left(p - \frac{3}{2}\right) - \frac{1}{2}} \tag{32}$$

Note that in final expression there are only poles of the low–pass filter; those due to the regularization procedure logically dissapeared [see lines 2–3 of (31)]. Given explicitely (numerically) the order p, the last formula may become more simple. For instance, for $p = 1$, the filter's poles are $z_{1,2} = \pm i$; formula (32) then gives

$$\mathrm{THD}[y(t,1)] \;=\; \sqrt{\frac{\pi^2}{3} - \pi \operatorname{cth}\pi} \;\approx\; 0.3695\,. \tag{33}$$

For $p = 2$, the filter's poles are $z_{1,2,3,4} = (\pm 1 \pm i)/\sqrt{2}$; formula (32) yields [see formula (34) at the bottom of the page 2485]. And so on...The THD computed accordingly to the formula (32) is plotted in Fig. 4. From the latter, we see that the dependency of the THD on the filter's order is very close to the exponential one (in logarithmic scale, the THD decrease practically linearly with p). The sawtooth wave, which is quite strongly distorted with respect to other standard periodic waveforms (THD is 80.3%), should be filtered by a third–order low–pass filter if we need the THD lesser than 10%, by a 6th–order filter if the THD should be lesser than 1%, by a 9th order–filter if the THD must be not greater than 0.1%, *etc.* One can roughy estimate that a third–order filter reduce the THD by a tenth.

D. THD of the Pulse Train Filtered by a Low–Pass Filter

Consider the pulse train (8) filtered by a pth order Butterworth filter whose transfer function is given by (27). We want to calculate the THD at the output of this filter as a function of the signal's duty cycle μ and of the filter's order p. This case is more sophisticated than previous two (it will require two different regularization procedures), but it is very interesting from practical point of view, since pulse train is a very frequent signal in digital electronics. From (8), it is clear that the absolute value of complex Fourier coefficients for this signal is:

$$\big|c_k[x]\big| \;=\; \frac{2A\,|\sin\pi\mu k|}{\pi k}\,, \qquad k \in \mathbb{N}^*\,.$$

After filtering and by putting again $\omega_c = \omega_0$ in (27), the output Fourier coefficients read:

$$\big|c_k[y]\big|^2 = \big|c_k[x]\,H(k\omega_0)\big|^2 = \frac{4\,A^2 H_0^2 \sin^2\pi\mu k}{\pi^2 k^2(1+k^{2p})}\,, \quad k \in \mathbb{N}^*\,.$$

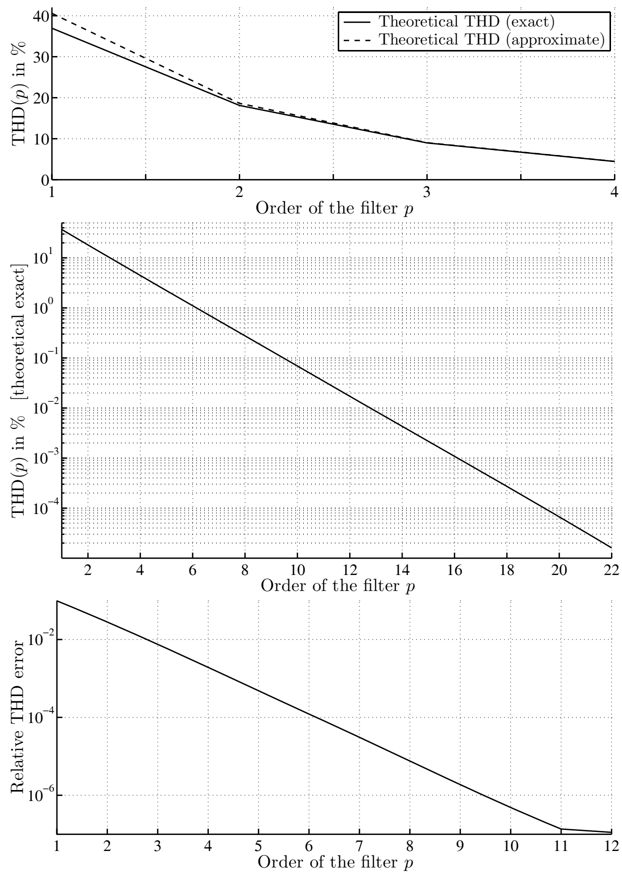

Fig. 4. Top and middle: Theoretical THD (exact and approximate) as a function of filter's order p computed accordingly to the analytcal residue–based method. Bottom: relative error between exact and approximate analytic methods. Nota bene: order of the filter is depicted as continuous variable only for more comfortable viewing.

By expression (18), the THD becomes:

$$\mathrm{THD}[y(t,\mu,p)] \;=\; \sqrt{\frac{2}{\sin^2\pi\mu}\sum_{n=2}^{\infty}\frac{\sin^2\pi\mu n}{n^2(1+n^{2p})}}\;. \qquad (35)$$

Thus, the problem of the closed–form evaluation of the THD is reduced to the analytical computation of the above series for $0<\mu<1$. Its calculation may be performed in the folowing way: we first remark that

$$\sum_{n=2}^{\infty}\frac{\sin^2\pi\mu n}{n^2(1+n^{2p})} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\frac{\sin^2\pi\mu n}{n^2(1+n^{2p})} - \frac{\sin^2\pi\mu}{2} - \frac{\pi^2\mu^2}{2}\,. \qquad (36)$$

Because the point zero for the summed function is not a pole but a removable singularity, the regularization was not needed at this stage. Now, according to the regularization procedures described in the appendix's sections B1–B2, we first introduce a small regularization parameter $\varepsilon\in\mathbb{R}$, and then, use formulæ (44) and (49). This gives[12] the formula (37) [at the bottom of the page 2486], in which points z_l are given by (30), except that $z_{2p+1}=\varepsilon$, and where we again denoted the part depending on ε by $g(\varepsilon,\mu,p)$:

$$g\,(\varepsilon,\mu,p)\;\equiv\;\lim_{z\to\varepsilon}\left[\frac{\cos\pi\big(z(2\mu-1)-2\mu\varepsilon\big)-\cos\pi z}{(1+z^{2p})\sin\pi z}\right]'_z\,.$$

The limit of $g\,(\varepsilon,\mu,p)$ when $\varepsilon\to 0$ exists and is equal to $2\mu\pi$. By calculating the value of residues at specified points, and by taking into account the convergence point of the above limit, the expression (37) becomes:

$$\sum_{n\in\mathbb{Z}}\frac{\sin^2\pi\mu n}{n^2(1+n^{2p})} = -\frac{\pi}{2}\sum_{s=1}^{2p}\left[\frac{\operatorname{ctg}\pi z_s}{z_s^2}\prod_{\substack{l=1\\ l\neq s}}^{2p}\frac{1}{z_s-z_l}\right]$$

$$+\frac{\pi}{2}\,\mathrm{Re}\sum_{s=1}^{2p}\left[\frac{e^{i\pi z_s(2\mu-1)}}{z_s^2\sin\pi z_s}\prod_{\substack{l=1\\ l\neq s}}^{2p}\frac{1}{z_s-z_l}\right]+\,\mu\pi^2\,.$$

By substituiting the last expression into (36), we finally arrive to the desired sum [see expression (38) at the bottom of the page 2486], from which the analytic expression for the THD is straightfoward [it is sufficient to put (38) into (35)]. In particular, for $p=1$, by (38), we simply have:

$$\sum_{n=2}^{\infty}\frac{\sin^2\pi\mu n}{n^2(1+n^2)} \;=\; \frac{\pi}{4}\left\{\frac{\operatorname{ch}\pi(2\mu-1)}{\operatorname{sh}\pi}-\operatorname{cth}\pi\right\}$$

$$+\frac{\pi^2\mu(1-\mu)}{2}-\frac{\sin^2\pi\mu}{2}\,. \qquad (39)$$

Consequently, $\mathrm{THD}[y(t,1)]\;=$

$$=\;\sqrt{\pi\frac{\operatorname{ch}\big(\pi(2\mu-1)\big)-\operatorname{ch}\pi}{2\cdot\operatorname{sh}\pi\cdot\sin^2\pi\mu}+\frac{\pi^2\mu(1-\mu)}{\sin^2\pi\mu}-1}\;.$$

Other particular cases may be without difficulty written in explicit form from the general formula (38). The implementation of this formula produces Fig. 5 as well as Tab. I (since 3–D plot may be difficult to read), which show the behaviour of the THD as a function of μ and p. It can be remarked that the THD of the output signal is symmetric about $\mu=0.5$, i.e.

$$\mathrm{THD}[y(t,\mu,p)]=\mathrm{THD}[y(t,1-\mu,p)]\,; \qquad (40)$$

[12]Notice that, in the first line of (37), we do not convert the unwanted second–order pole into two first–order ones which are both "well–located" (as we did in previous section when introducing $\widetilde{f}$ instead of f), but into a second–order pole which is simply displaced from the unwanted location (this permits to simultaneously shift the argument of the sinus). The rest of the calculation in (37) is very similar to (31) and what follows, except that for the first sum we use (44), while for the second one we employ (49) [see appendix for details].

$$\mathrm{THD}[y(t,2)]\;=\;\sqrt{\pi\,\frac{\operatorname{ctg}\dfrac{\pi}{\sqrt2}\operatorname{cth}^2\dfrac{\pi}{\sqrt2}-\operatorname{ctg}^2\dfrac{\pi}{\sqrt2}\operatorname{cth}\dfrac{\pi}{\sqrt2}-\operatorname{ctg}\dfrac{\pi}{\sqrt2}-\operatorname{cth}\dfrac{\pi}{\sqrt2}}{\sqrt2\left(\operatorname{ctg}^2\dfrac{\pi}{\sqrt2}+\operatorname{cth}^2\dfrac{\pi}{\sqrt2}\right)}+\frac{\pi^2}{3}-1}\;\approx\;0.1811 \qquad (34)$$

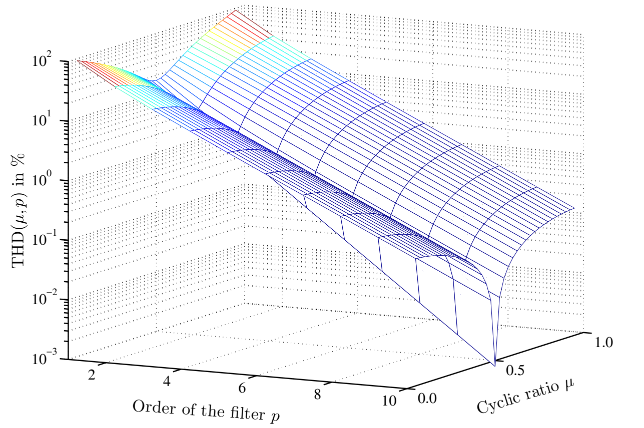

Fig. 5. Theoretical THD as a function of cyclic ratio (duty cycle) μ and of filter's order p computed accordingly to the analytcal residue–based method.

that is to say, the signal keeps this property after filtering as well. In logarithmic scale, the THD decrease almost linearly with p; in addition, the closer the pulse train to the square wave, the faster the decresing of the THD with the order p (i.e. $\mu = 0.5$ represents the line of the fastest descent and of the lesser THD for given p).

IV. Conclusions

We presented a fully analytic method for the calculation of the THD, which is an important performance criterion for almost any communication device. Signal, whose THD is computed, is modeled as a T–periodic $L^2(T)$ signal, passed through L^1 filter with partially known transfer function (this corresponds quite well to a typical transmitter output stage). The calculation of the THD, which depends on the Fourier coefficients of the filtered signal, is usually reduced to that of some infinite sums. In contrast to classic methods, which evaluate THD sums empirically or numerically, the proposed one is completely analytic, quite general and gives the closed–form expression for the THD. Mathematical essence of the method, which is based on the theory of functions of a complex variable, is described in the appendix. We rigorously carry out calculations, showing that the method could be practically always employed for the proposed and similar aims. We also considered practical examples including band–pass and low–pass filters (typical filters at the output of the power amplifiers) with three frequent periodic signals. In particular, these exemples showed that, in the case of band–pass filters with quality factor Q, for large Q, the THD is inversely proportional to Q, and in the case of pth–order low–pass filters, for large p, the THD decrease exponentially with p. Under specified conditions, which are very mild, the proposed method can be similarly applied to other filters and signals. However, it should be noted that as the filter's topology and the signal's shape becomes more and more sophisticated, calculations becomes more and more long and complicated. Thus, it may be quite laborious to establish the closed–form expression of the THD for some type of signals and filters, and one should estimate the tradeoff between the need of the analytic result and the work to be done. Note finally that the proposed method may be used for discrete–time signals as well, in which case Fourier coefficients have to be replaced by DFT ones and the residue technique for the summation of series should be applied to finite sums.

Appendix
Cauchy Method of Residues for the Summation of Series

A. General Theory

The fundamental theorem of the theory of functions of a complex variable is the Cauchy theorem of residues.[13] This theorem states that for the function $f(z)$, which is analytic[14] and single–valued inside and on a simple closed curve γ except possibly for a finite number of isolated singularities,

[13] As to the theory of functions of a complex variable, the readers are referred to these classic books: [27], [28], [30], [75]–[88].

[14] In complex analysis, terms "analytic", "regular" and "holomorphic" are synonyms.

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\frac{\sin^2\pi\mu n}{n^2(1+n^{2p})} &= \lim_{\varepsilon\to 0}\left\{\frac{1}{2}\sum_{n\in\mathbb{Z}}\frac{1}{(n-\varepsilon)^2(1+n^{2p})}-\frac{1}{2}\,\mathrm{Re}\sum_{n\in\mathbb{Z}}\frac{e^{2i\pi\mu(n-\varepsilon)}}{(n-\varepsilon)^2(1+n^{2p})}\right\}\\ &= \lim_{\varepsilon\to 0}\left\{-\frac{\pi}{2}\sum_{l=1}^{2p+1}\operatorname*{res}_{z=z_l}\frac{\operatorname{ctg}\pi z}{(z-\varepsilon)^2(1+z^{2p})}+\frac{\pi}{2}\,\mathrm{Re}\sum_{l=1}^{2p+1}\operatorname*{res}_{z=z_l}\frac{e^{i\pi z(2\mu-1)}e^{-2i\pi\mu\varepsilon}}{(z-\varepsilon)^2(1+z^{2p})\sin\pi z}\right\}\\ &= -\frac{\pi}{2}\sum_{l=1}^{2p}\operatorname*{res}_{z=z_l}\frac{\operatorname{ctg}\pi z}{z^2(1+z^{2p})}+\frac{\pi}{2}\,\mathrm{Re}\sum_{l=1}^{2p}\operatorname*{res}_{z=z_l}\frac{e^{i\pi z(2\mu-1)}}{z^2(1+z^{2p})\sin\pi z}+\frac{\pi}{2}\lim_{\varepsilon\to 0}g\left(\varepsilon,\mu,p\right),\end{aligned}\tag{37}$$

$$\sum_{n=2}^{\infty}\frac{\sin^2\pi\mu n}{n^2(1+n^{2p})} = -\frac{\pi}{4}\sum_{s=1}^{2p}\left[\frac{\operatorname{ctg}\pi z_s}{z_s^2}\prod_{\substack{l=1\\ l\neq s}}^{2p}\frac{1}{z_s-z_l}\right]+\frac{\pi}{4}\,\mathrm{Re}\sum_{s=1}^{2p}\left[\frac{e^{i\pi z_s(2\mu-1)}}{z_s^2\sin\pi z_s}\prod_{\substack{l=1\\ l\neq s}}^{2p}\frac{1}{z_s-z_l}\right]+\frac{\pi^2\mu(1-\mu)}{2}-\frac{\sin^2\pi\mu}{2}\tag{38}$$

TABLE I

THEORETICAL THD IN % AS A FUNCTION OF THE CYCLIC RATIO μ AND OF THE FILTER'S ORDER p COMPUTED ACCORDINGLY TO THE ANALYTICAL RESIDUE–BASED METHOD. THE VALUES OF THE THD FOR $\mu = 0.6, 0.7, 0.8, 0.9$ MAY BE EASILY OBTAINED FROM THE ABOVE ONES SINCE THE THD IS SYMMETRICAL ABOUT $\mu = 0.5$ [SEE E.G. (40)]. CASE $p = 0$ CORRESPONDS TO THE NON–FILTERED SIGNAL, WHOSE THD IS GIVEN BY (17) AND IS DEPICTED IN FIG. 2.

p	$\mu = 0.1$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = 0.5$
0	191.0	113.3	76.37	55.62	48.34
1	80.04	57.74	39.03	23.81	16.35
2	36.26	29.10	20.38	11.32	5.348
3	17.40	14.48	10.34	5.555	1.760
4	8.539	7.200	5.191	2.753	$5.837 \cdot 10^{-1}$
5	4.233	3.587	2.597	1.370	$1.942 \cdot 10^{-1}$
6	2.108	1.790	1.298	$6.839 \cdot 10^{-1}$	$6.469 \cdot 10^{-2}$
7	1.052	$8.945 \cdot 10^{-1}$	$6.494 \cdot 10^{-1}$	$3.416 \cdot 10^{-1}$	$2.155 \cdot 10^{-2}$
8	$5.257 \cdot 10^{-1}$	$4.470 \cdot 10^{-1}$	$3.247 \cdot 10^{-1}$	$1.707 \cdot 10^{-1}$	$7.185 \cdot 10^{-3}$
9	$2.627 \cdot 10^{-1}$	$2.234 \cdot 10^{-1}$	$1.623 \cdot 10^{-1}$	$8.536 \cdot 10^{-2}$	$2.395 \cdot 10^{-3}$
10	$1.313 \cdot 10^{-1}$	$1.117 \cdot 10^{-1}$	$8.117 \cdot 10^{-2}$	$4.268 \cdot 10^{-2}$	$7.983 \cdot 10^{-4}$
11	$6.567 \cdot 10^{-2}$	$5.586 \cdot 10^{-2}$	$4.058 \cdot 10^{-2}$	$2.133 \cdot 10^{-2}$	$2.660 \cdot 10^{-4}$
12	$3.283 \cdot 10^{-2}$	$2.793 \cdot 10^{-2}$	$2.029 \cdot 10^{-2}$	$1.066 \cdot 10^{-2}$	$8.853 \cdot 10^{-5}$
13	$1.641 \cdot 10^{-2}$	$1.396 \cdot 10^{-2}$	$1.014 \cdot 10^{-2}$	$5.334 \cdot 10^{-3}$	$2.916 \cdot 10^{-5}$
14	$8.209 \cdot 10^{-3}$	$6.983 \cdot 10^{-3}$	$5.073 \cdot 10^{-3}$	$2.667 \cdot 10^{-3}$	$9.064 \cdot 10^{-6}$

the contour integral

$$\oint\limits_{\gamma} f(z)\,dz = \begin{cases} 0\,, & \text{if } f(z) \text{ has no singularities inside } \gamma\,, \\[2mm] 2\pi i \displaystyle\sum_{l=1}^{m} \operatorname*{res}_{z=z_l} f(z)\,, & \text{otherwise}\,, \end{cases}$$

where $\{z_l\}_{l=1}^{m}$ are the isolated singularities of the function $f(z)$ enclosed by the contour γ. We recall that singularities are special points at which function is not regular. For single–valued functions, singularities are usually classified as follows: removable singularities, poles and isolated essential singularities. Note that in practice, function $f(z)$ is often a meromorphic function (it is especially true for the functions that we face when computing the THD); in this case, singularities $\{z_l\}_{l=1}^{m}$ are poles of the function $f(z)$.

We now describe the method of residues for the summation of series. Unfortunately, this method is present only in a small amount of good complex variable literature [75, pp. 188–191], [85, pp. 261–269], [76], [51, pp. 69–70], [86, pp. 115–116], [87, pp. 123–125], and it is just mentioned in [30, pp. 114–115], [27, p. 205]. Besides, considered sums, integrals, and proofs vary significantly depending on the authors and on their objectives. We will therefore try to present the method in a well–structured and concise way adjusted to our purposes. Let $f(z)$, in addition to the aforementioned conditions, have no singularities at integers and be bounded in the following way: as $|z| \to \infty$,

$$\bigl|f(z)\bigr| \leqslant \frac{C}{|z|^{\xi}}\,, \quad \xi > 1\,, \quad C = \text{const}\,. \tag{41}$$

A typical example of such a function may be a rational function for which $|f(z)| = O(|z|^{-s})$, $s = 2, 3, 4, \ldots$, as $|z| \to \infty$, and which does not have poles at integers. Consider now the integral

$$\oint\limits_{\gamma_R} f(z)\,\operatorname{ctg}(\pi z)\,dz\,, \tag{42}$$

where contour γ_R is given by the circle $|z| = R + \alpha$, with $R \to \infty$, $R \in \mathbb{N}$ and $\alpha \in (0, 1)$.[15] On the one hand, by the residue theorem we have:

$$\begin{aligned} &\lim_{R\to\infty} \oint\limits_{\gamma_R} f(z)\,\operatorname{ctg}(\pi z)\,dz \qquad (43) \\ &= 2\pi i \left\{ \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} f(n) + \sum_{l=1}^{m} \operatorname*{res}_{z=z_l} \bigl[f(z)\,\operatorname{ctg}(\pi z)\bigr] \right\}\,, \end{aligned}$$

where $\{z_l\}_{l=1}^{m}$ are all finite singularities of the function $f(z)$ in the complex plane, and where we took into account that the poles of $\operatorname{ctg}(\pi z)$ are simple and occur at $z = n$, $n \in \mathbb{Z}$. On the other hand, as $R \to \infty$, $R \in \mathbb{N}$, the integral (42) approaches zero:

$$\begin{aligned} &\lim_{R\to\infty} \left| \oint\limits_{\gamma_R} f(z)\,\operatorname{ctg}(\pi z)\,dz \right| \\ &\leqslant 2\pi \lim_{R\to\infty} \Bigl[(R+\alpha) \max_{z\in\gamma_R} \bigl|f(z)\,\operatorname{ctg}(\pi z)\bigr|\Bigr] \\ &\leqslant 2\pi C \lim_{R\to\infty} \frac{\max\limits_{z\in\gamma_R} |\operatorname{ctg}(\pi z)|}{R^{\xi-1}} = 0\,, \end{aligned}$$

[15] A somewhat different method is presented in [75] and [30], where the considered integrand is the same (with additional π), but the integral is taken around the square which intersects coordinate axes at half–integers and whose perimeter tends to infinity [by the way, the readers of this book should beware of misprints, e.g. p. 189, $2\pi i$ is forgotten in the right part of equation (1)].

since $\xi > 1$ and $\operatorname{ctg}(\pi z)$ may be always bounded by a constant on the contour γ_R (thanks to the small constant α which is strictly non–integer).[16] Whence, it follows immediately that

$$\sum_{n=-\infty}^{+\infty} f(n) = -\pi \sum_{l=1}^{m} \operatorname*{res}_{z=z_l} \left[f(z) \operatorname{ctg}(\pi z) \right] . \qquad (44)$$

This equation is the key formula for the evaluation of infinite sums. By the way, the condition (41) on the summed function $f(z)$ can be replaced by a more general one:

$$|z f(z) \operatorname{ctg} \pi z| \to 0 \qquad \text{as } |z| \to \infty \,, \ |z| \notin \mathbb{N}^* \,, \qquad (45)$$

which may be sometimes more convenient. Note that both conditions (41) and (45) are closely related to the chosen contour γ_R, around which the integral (42) is taken, and on which $\operatorname{ctg}(\pi z)$ should be bounded. Namely, the integration contour was chosen so that the line integral (42) vanishes.

Application of the same technique to slightly different line integral provides also this useful formula:

$$\sum_{n=-\infty}^{+\infty} (-1)^n f(n) = -\pi \sum_{l=1}^{m} \operatorname*{res}_{z=z_l} \left[\frac{f(z)}{\sin \pi z} \right] . \qquad (46)$$

It may be of interest to note that the formula (46) may be used not only under conditions (41) or (45), but also under more weak requirement:

$$\left| \frac{z f(z)}{\sin \pi z} \right| \to 0 \qquad \text{as } |z| \to \infty \,, \ |z| \notin \mathbb{N}^* \,. \qquad (47)$$

This condition is less restrictive than conditions (41) and (45), and it can be particularly useful when the summed function is not rational and contains, for example, trigonometric functions. Consider for instance $f(z) = e^{i\alpha z} g(z)$, where α is a real constant and $g(z)$ is a rational function such that $|g(z)| = O(|z|^{-2})$ as $|z| \to \infty$. For such a summed function, condition (47) is verified for $-\pi < \alpha < \pi$, while (41) and (45) are never verified. Same results are obtained if encountering $\sin \alpha z$ or $\cos \alpha z$ instead of $e^{i\alpha z}$ in $f(z)$.

A short demonstration of the practical use of the method may be quite fruitful at this stage. Suppose that one wants to evaluate the following series:

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n \sin \alpha n}{(n-\varepsilon)^2} \,, \qquad -\pi < \alpha < \pi \,, \quad \varepsilon \notin \mathbb{Z} \,.$$

Summed function satisfies condition (47); consequently, formula (46) may be used for the analytic computation of this series. The latter yields:

$$\sum_{n=-\infty}^{+\infty} \frac{(-1)^n \sin \alpha n}{(n-\varepsilon)^2} = -\pi \operatorname*{res}_{z=\varepsilon} \left[\frac{\sin \alpha z}{(z-\varepsilon)^2 \sin \pi z} \right] \qquad (48)$$

$$= -\pi \lim_{z\to\varepsilon} \left[\frac{\sin \alpha z}{\sin \pi z} \right]'_z = \frac{\pi \sin \alpha\varepsilon}{\sin \pi\varepsilon} \Big\{ \pi \operatorname{ctg} \pi\varepsilon - \alpha \operatorname{ctg} \alpha\varepsilon \Big\} .$$

Finally, the use of the residues for the summation of series may be also extended to the finite sums. For example, if contour γ encloses the points $z = K, z = K+1, \ldots, z = N$, where $K \in \mathbb{Z}$ and $N \in \mathbb{Z}$, and $f(z)$ is analytic and single–valued within and on γ except for a finite number of isolated singularities $\{z_l\}_{l=1}^{m}$, none of which are integer, then:

$$\sum_{n=K}^{N} f(n) = \frac{1}{2i} \oint\limits_{\gamma} f(z) \operatorname{ctg}(\pi z) \, dz - \pi \sum_{l=1}^{m} \operatorname*{res}_{z=z_l} \left[f(z) \operatorname{ctg}(\pi z) \right].$$

It is frequently possible to chose the path γ so that the integral on the right vanishes. For example, by slightly changing the above integrand, and by taking the integration contour γ as a rectangle with vertices at $[2\pi + ia, ia, -ia, 2\pi - ia]$, $a \to \infty$, one may obtain the following summation formula:

$$\sum_{n=0}^{N-1} f\left(\frac{2n\pi}{N}\right) = -\frac{N}{2} \sum_{l=1}^{m} \operatorname*{res}_{z=z_l} \left[f(z) \operatorname{ctg}\left(\frac{Nz}{2}\right) \right] ,$$

where $f(z)$ is the rational function of $\sin z$ or/and of $\cos z$ none of whose poles lie on γ or on the coordinate axes, where z_l are poles of the function $f(z)$ inside γ, and where $|f(z)| \to 0$ as $\operatorname{Im}[z] \to \pm\infty$. This formula may be particularly appreciated for the analytic computation of the discrete Fourier, cosine and Hartley transforms (DFT, DCT & DHT). More details on the application of the residue method for the summation of finite series may be found in [85, p. 262].

B. Special Cases—Regularization

Some of the restrictions on the function $f(z)$ may be quite annoying, but in many cases they can be circumvented. The most common cases, when one of these requirements is not satisfied, are these two. First, function $f(z)$ may have poles at integers, and consequently, it may be not regular at $z \in \mathbb{Z}$. In this case, it is often possible to regularize the problem by introducing a small constant ε into the function $f(z)$, in such way that the unwanted pole be slightly displaced. The formula (44) may be therefore used by an appropriating limiting procedure [if the limit on the right for $\varepsilon \to 0$ exists]. Second, summed function may fulfill (41) or (45) only partially; e.g. it may be satisfied only on the real or imaginary axis, but not on the both as required. In this case, the problem can be often regularized either by trying to make use of different integrands in (42) [e.g. by trying to use formula (46) instead of (44)], or by an appropriate choice of the integration contour, or both at the same time. Examples below permit to better understand how these procedures may be performed in practice.

1) Summed Function has Poles at Integers: We wish to evaluate the following series:

$$\sum_{n\in\mathbb{N}^*} \frac{1}{n^2} \,.$$

The result is well known from analysis[17]; it is equal to $\zeta(2) = \pi^2/6$, but we want to verify it by using the Cauchy method of residues. Let $f(z) = z^{-2}$. This function satisfies all requirements for the application of the formula (44), except that it is not regular at $z = 0$; this point is a double pole for

[16] More generally, function $\operatorname{ctg}(\pi z)$ is bounded in the whole complex plane, except discs $|z - n| < \varepsilon$, where $n \in \mathbb{Z}$ and ε is an arbitrary small positive constant [76], [78], [82].

[17] More precisely, analytically, it was first computed by L. Euler in 1735 (the problem of the evaluation of this series is sometimes referred as *Basel problem*).

$f(z)$. We therefore try to regularize the problem by adding a small real positive constant ε^2 to its denominator:

$$\widetilde{f}(z) = \frac{1}{z^2+\varepsilon^2}\,, \qquad \varepsilon \in \mathbb{R}\,.$$

Function $\widetilde{f}(z)$ is now regular for $z \in \mathbb{Z}$, and also for $z \in \mathbb{R}$, since it has two poles at points $\pm i\varepsilon$. Obviously, as $\varepsilon \to 0$, function $\widetilde{f}(z) \to f(z)$. The application of the formula (44) is now possible; it reads:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{\varepsilon\to 0} \sum_{n=1}^{\infty} \frac{1}{n^2+\varepsilon^2} = \frac{1}{2} \lim_{\varepsilon\to 0} \left\{ \sum_{n=-\infty}^{+\infty} \frac{1}{n^2+\varepsilon^2} - \frac{1}{\varepsilon^2} \right\}$$

$$= -\frac{\pi}{2} \lim_{\varepsilon\to 0} \left\{ \operatorname*{res}_{z=i\varepsilon} \frac{\operatorname{ctg}\pi z}{z^2+\varepsilon^2} + \operatorname*{res}_{z=-i\varepsilon} \frac{\operatorname{ctg}\pi z}{z^2+\varepsilon^2} + \frac{1}{\pi\varepsilon^2} \right\}$$

$$= \frac{\pi}{2} \lim_{\varepsilon\to 0} \left\{ \frac{\operatorname{cth}\pi\varepsilon}{\varepsilon} - \frac{1}{\pi\varepsilon^2} \right\} = \frac{\pi^2}{6}\,,$$

since the last limit converges to $\pi/3$.

2) Summed Function Satisfies Condition (41) *Only Partially:* As we previously noticed, if the summed function contains trigonometric or exponential functions in the numerator, e.g.:

$$f_1(k) = \frac{\sin \alpha k}{k^2+1}\,, \quad f_2(k) = \frac{e^{i\alpha k}}{(k-\ln 2)^2}\,, \qquad \alpha \in \mathbb{R}\,,$$

formula (44) cannot be applied, because conditions (41) and (45) are satisfied for them either only for $\operatorname{Re}[z] \to \pm\infty$, or for $\operatorname{Im}[z] \to \pm\infty$, or for three of them, but not for $|z| \to \infty$. However, in many cases, sums of such series can be still computed via an appropriate use of the formula (46) instead of (44). On the one hand, the condition (47) is less restrictive than (41) and (45) and the summed functions f_1 and f_2 satisfy it for $-\pi < \alpha < \pi$. On the other hand, the formula (46) is suitable for alternating series, while we face an "ordinary" series. The regularization, in this case, consists in reducing an "ordinary" series to an alternating one. In view of the fact that $(-1)^n = e^{\pm i\pi n}$, formula (46) may be rewritten as

$$\sum_{n=-\infty}^{+\infty} f(n) \;=\; -\pi \sum_{l=1}^{m} \operatorname*{res}_{z=z_l} \left[\frac{f(z)e^{\pm i\pi z}}{\sin \pi z} \right]. \tag{49}$$

It is important to note here that functions $\sin(\alpha z)g(z)e^{\pm i\pi z}$ and $\cos(\alpha z)g(z)e^{\pm i\pi z}$ do not satisfies (47), while $e^{i\alpha z}g(z)e^{\pm i\pi z}$ still does in some range of α. Namely, α should be such that $-\pi < \alpha \pm \pi < \pi$; consequently, in (49) we choose sign "−" if $0 < \alpha < 2\pi$ and "+" if $-2\pi < \alpha < 0$. Hence, series, containing trigonometric functions and which are not alternating, should be computed via exponential series [i.e. they cannot be computed directly as we did in (48)].

For example, we wish to calculate two following series:

$$\sum_{n\in\mathbb{Z}} \frac{\sin \alpha n}{(n-\varepsilon)^2}\,, \qquad \sum_{n\in\mathbb{Z}} \frac{\cos \alpha n}{(n-\varepsilon)^2}\,, \tag{50}$$

where $0 < \alpha < 2\pi$ and $\varepsilon \notin \mathbb{Z}$. To this end, we must first compute the following series:

$$\sum_{n\in\mathbb{Z}} \frac{e^{i\alpha n}}{(n-\varepsilon)^2}\,. \tag{51}$$

The sign in (49) should be therefore "−", and thus, this formula yields:

$$\sum_{n=-\infty}^{+\infty} \frac{e^{i\alpha n}}{(n-\varepsilon)^2} \;=\; -\pi \operatorname*{res}_{z=\varepsilon} \left[\frac{e^{i(\alpha-\pi)z}}{(z-\varepsilon)^2 \sin \pi z} \right] \tag{52}$$

$$= -\pi \lim_{z\to\varepsilon} \left[\frac{e^{i(\alpha-\pi)z}}{\sin \pi z} \right]'_z = \frac{\pi e^{i\varepsilon(\alpha-\pi)}}{\sin \pi\varepsilon} \Big\{ i(\pi-\alpha) + \pi \operatorname{ctg} \pi\varepsilon \Big\}\,.$$

On taking imaginary parts, one can deduce the summation formula containing sinus:

$$\sum_{n=-\infty}^{+\infty} \frac{\sin \alpha n}{(n-\varepsilon)^2} \;=\; \frac{\pi \sin \varepsilon(\alpha-\pi)}{\sin \pi\varepsilon}$$

$$\cdot \Big\{ (\pi-\alpha) \operatorname{ctg} \varepsilon(\alpha-\pi) + \pi \operatorname{ctg} \pi\varepsilon \Big\}\,. \tag{53}$$

Analogously, by equating real parts of (52), one can obtain:

$$\sum_{n=-\infty}^{+\infty} \frac{\cos \alpha n}{(n-\varepsilon)^2} \;=\; \frac{\pi \cos \varepsilon(\alpha-\pi)}{\sin \pi\varepsilon}$$

$$\cdot \Big\{ -(\pi-\alpha) \operatorname{tg} \varepsilon(\alpha-\pi) + \pi \operatorname{ctg} \pi\varepsilon \Big\}\,. \tag{54}$$

It is easy to verify by numerical summation that these formulæ are correct, while direct application of (49) to the series (50) [i.e. not to (51)] would produce incorrect results. By the way, since many of the formulæ obtained throughout the paper are quite long, in order to avoid any error, they were all carefully verified, including by numeric summation and by computer simulations[18].

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